Derivation of a transfer function model for a high pressure pipeline

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Abstract

In this report a lumped transfer function model for High Pressure Natural Gas Pipelines is derived. Starting with a partial nonlinear differential equation (PDE) model a high order continuous state space (SS) linear model is obtained using a finite difference method. Next, from the SS representation an infinite order transfer function (TF) model is calculated. In the end, this TF is approximated by a compact non-rational function.

1 Introduction

In this report we investigate the problem of the representation of a high pressure gas pipeline by a compact non rational transfer function model. This model is used to simulate mass flow and pressure in a small high pressure pipeline, and although this is a simple model with few parameters, it seems to have an accuracy comparable to the SIMONE® simulator. Since this kind of models are suitable to control design and are well understood by control practitioners, it is our intention to apply them to gas leakage detection and gas network control.

2 STATE-SPACE DISCRETE-IN-SPACE MODEL

The gas dynamics within the pipes is represented by a set of partial differential equations (PDE). If we neglect the viscous and the turbulent effects of the flow and assume small temperature changes within the gas and small heat exchanges with the surroundings of the pipeline, it can be described by the one-dimensional hyperbolic model

$$
\begin{cases}
\dfrac{\partial q(\ell,t)}{\partial t} = -\mathcal{A}\dfrac{\partial p(\ell,t)}{\partial \ell} - \dfrac{f_c c^2}{2\mathcal{D}\mathcal{A}}\dfrac{q^2(\ell,t)}{p(\ell,t)} \\[2ex]
\dfrac{\partial p(\ell,t)}{\partial t} = -\dfrac{c^2}{\mathcal{A}}\dfrac{\partial q(\ell,t)}{\partial \ell},
\end{cases}
\tag{1}
$$

where ℓ is space, t is time, p is edge pressure-drop, q is mass flow, $\mathcal{A}$ is the cross-sectional area, $\mathcal{D}$ is the pipe diameter, c is the isothermal speed of sound, and f_c is the friction factor.

In this research we linearised model (1) around the operational levels $(p_m(\ell), q_m)$, where we assume a constant flow rate, and from the first equation of (1)

$$
p_m(\ell) = \sqrt{p_m^2(\ell_0) - \frac{f_c c^2}{2\mathcal{D}\mathcal{A}^2}q_m^2(\ell - \ell_0)}.
$$

Hence we set $p(\ell,t) = p_m(\ell) + \Delta p(\ell,t)$ and $q(\ell,t) = q_m + \Delta q(\ell,t)$, where $\Delta p(\ell,t)$ and $\Delta q(\ell,t)$ are deviations from the pressure/flow operational levels, respectively. Then $\dfrac{q^2(\ell,t)}{p(\ell,t)} = \dfrac{(q_m + \Delta q(\ell,t))^2}{p_m(\ell) + \Delta p(\ell,t)} = \dfrac{q_m^2}{p_m(\ell)} + 2\dfrac{q_m}{p_m(\ell)}\Delta q(\ell,t) - \dfrac{q_m^2}{p_m^2(\ell)}\Delta p(\ell,t)$

The third term may be neglected since the distribution networks operate at very high pressure, ca. 80 bar. Then we substitute the remaining in the first equation

$$
\frac{\partial q(\ell,t)}{\partial t} = -\mathcal{A}\frac{\partial p(\ell,t)}{\partial \ell} - \frac{f_c c^2}{2\mathcal{D}\mathcal{A}}\frac{q_m}{p_m}\left(q_m + 2\Delta q(\ell,t)\right).
$$

Assuming small oscillations, $\Delta q(\ell,t) \approx 2\Delta q(\ell,t)$, we may have $(q_m + 2\Delta q(\ell,t)) \approx q(\ell,t)$ and obtain the following linearized model:

$$
\begin{cases}
\dfrac{\partial q(\ell,t)}{\partial t} = -\mathcal{A}\dfrac{\partial p(\ell,t)}{\partial \ell} - 2\alpha q(\ell,t) \\[2ex]
\dfrac{\partial p(\ell,t)}{\partial t} = -\dfrac{c^2}{\mathcal{A}}\dfrac{\partial q(\ell,t)}{\partial \ell}.
\end{cases}
\tag{2}
$$

where

$$\alpha = \frac{f_c c^2}{4 \mathcal{D} \mathcal{A}} \frac{q_m}{p_m}. \tag{3}$$

Next, decompose the pipeline into sections $\mathcal{L}_i = [\ell_{i-1}, \ell_i]$, $i = 1, 2, \ldots, N$, where $\ell_0 = 0$, $\ell_N = L$ and L is the length of the pipeline. We assume the massflow to be the same in each section and accordingly define the following notation:

$$\begin{aligned}
q_0(t) &= q(0, t) \\
q_i(t) &= q(\ell, t), \quad \ell_{i-1} < \ell < \ell_i, \quad i = 1, 2, \ldots, N \\
q_{N+1}(t) &= q(L, t) \\
p_i(t) &= p(\ell_i, t), \quad i = 0, 1, \ldots, N.
\end{aligned} \tag{4}$$

Making

$$\left. \frac{\partial \cdot (\ell, t)}{\partial \ell} \right|_{\ell = \ell_i} \approx \frac{\cdot(\ell_i, t) - \cdot(\ell_{i-1}, t)}{\ell_i - \ell_{i-1}}, \quad i = 1, 2, \ldots, N, \tag{5}$$

we can now approximate the linearized PDE (2) by the following discrete-in-space model

$$\begin{aligned}
\dot{q}_i(t) &= \frac{\mathcal{A}}{\Delta \ell} [p_{i-1}(t) - p_i(t)] - 2\alpha q_i(t), \\
&\qquad\qquad i = 1, \ldots, N \\
\dot{p}_{j-1}(t) &= \frac{c^2}{\mathcal{A}\Delta \ell} [q_{j-1}(t) - q_j(t)] \\
&\qquad\qquad j = 1, \ldots, N+1.
\end{aligned} \tag{6}$$

where

$$\Delta \ell = \ell_{i+1} - \ell_i = \frac{L}{N}, \quad i = 0, \ldots, N-1. \tag{7}$$

The pipe can then be described by the following state-space model:

$$\begin{aligned}
\dot{x}_1(t) &= -\frac{c^2}{\mathcal{A}\Delta \ell} x_{N+2}(t) + \frac{c^2}{\mathcal{A}\Delta \ell} u_1(t) \\
\dot{x}_i(t) &= \frac{c^2}{\mathcal{A}\Delta \ell} x_{N+i}(t) - \frac{c^2}{\mathcal{A}\Delta \ell} x_{N+i+1}(t) \\
\dot{x}_{N+1}(t) &= \frac{c^2}{\mathcal{A}\Delta \ell} x_{2N+1}(t) - \frac{c^2}{\mathcal{A}\Delta \ell} u_2(t) \\
\dot{x}_{N+1+j}(t) &= \frac{\mathcal{A}}{\Delta \ell} x_j(t) - \frac{\mathcal{A}}{\Delta \ell} x_{j+1}(t) - 2\alpha x_{N+1+j}(t) \\
y_1(t) &= x_1(t) \\
y_2(t) &= x_{N+1}(t),
\end{aligned} \tag{8}$$

where $i = 1, \ldots, N$, $j = 1, \ldots N$ and also

$$\begin{aligned}
u(t) &= \begin{bmatrix} q_0(t) & q_{N+1}(t) \end{bmatrix}^T = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T \\
x(t) &= \begin{bmatrix} p_0(t) & \cdots & p_N(t) & | & q_1(t) & \cdots & q_N(t) \end{bmatrix}^T \\
y(t) &= \begin{bmatrix} p_0(t) & p_N(t) \end{bmatrix}^T = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T.
\end{aligned} \tag{9}$$

In matrix notation:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t).
\end{aligned} \tag{10}$$

Partition A as:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \tag{11}$$

where

$$A_{11} = 0_{(N+1)\times(N+1)} \tag{12}$$

$$A_{12} = \begin{bmatrix} -\dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 & \cdots & 0 & 0 \\ \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} \\ 0 & 0 & \cdots & 0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} \end{bmatrix} \in \mathbb{R}(N+1) \times N \tag{13}$$

$$A_{21} = \begin{bmatrix} \dfrac{\mathcal{A}}{\Delta\ell} & -\dfrac{\mathcal{A}}{\Delta\ell} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{\mathcal{A}}{\Delta\ell} & -\dfrac{\mathcal{A}}{\Delta\ell} \end{bmatrix} \in \mathbb{R}^{N\times(N+1)} \tag{14}$$

$$A_{22} = -2\alpha I_N \tag{15}$$

$$B = \frac{c^2}{\mathcal{A}\Delta\ell}\left[\begin{array}{c|c} e_1 & -e_{N+1} \end{array}\right] \tag{16}$$

$$C = \left[\begin{array}{c|c} e_1 & e_{N+1} \end{array}\right]^T \tag{17}$$

where e_i is the ith vector of the canonical orthonormal basis, i.e., a vector with the the ith component equal to one and the others equal to zero.

3 Spectral analysis of A

In order to learn more about the system (3), we analyse the spectrum of matrix A. Therefore the following theorem:

Theorem 1 *The eigenvalues of A defined in* (12)–(15) *are*

$$\lambda_0 = 0 \tag{18}$$

$$\lambda_{\pm k} = -\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m} \pm j\sqrt{\left(2\frac{c}{\Delta\ell}\sin\left(\frac{k\pi}{2(N+1)}\right)\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2}, \quad k = 1,\ldots,N \tag{19}$$

Proof: In the Appendix A it was proven that matrix $\bar{A}$ is equivalent to A up to a similarity transformation. Consequently they have the same eigenvalues and using (124), we have:

$$\det(sI_{2N+1} - A) = \det(sI_{2N+1} - \bar{A}) = s\det\left[\begin{array}{c|c} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{array}\right] = 0$$

And this is equivalent to $s = 0$ and $\det\left[\begin{array}{c|c} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{array}\right] = 0$. Therefore, A has a zero eigenvalue, that is, $\lambda_0 = 0$.

From Fact 2.13.10 in [1, pp. 62–63], we have that for arbitrary matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D} \in \mathbb{R}^{N \times N}$ such that $\mathbf{AB} = \mathbf{BA}$ then

$$\det\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \det(\mathbf{DA} - \mathbf{CB}).$$

Thus, if we take

$$\begin{aligned} \mathbf{A} &= sI_N - \bar{A}_{11} \\ \mathbf{B} &= -\bar{A}_{12} \\ \mathbf{C} &= -\bar{A}_{21} \\ \mathbf{D} &= sI_N - \bar{A}_{22} \end{aligned}$$

we see that $(sI_N - \bar{A}_{11})(-\bar{A}_{12}) = (-\bar{A}_{12})(sI_N - \bar{A}_{11}) \Rightarrow \mathbf{AB} = \mathbf{BA}$ because $sI_N - \bar{A}_{11}$ is a diagonal matrix. Consequently

$$\det\left[\begin{array}{c|c} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{array}\right] = \det\left((sI_N - \bar{A}_{22})(sI_N - \bar{A}_{11}) - \bar{A}_{21}\bar{A}_{12}\right).$$

Given that $\bar{A}_{11} = 0_{N \times N}$ and $\bar{A}_{22} = -2\alpha I_N$, then

$$\det\left[\begin{array}{c|c} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{array}\right] = \det\left((s^2 + 2\alpha s)I_N - \bar{A}_{21}\bar{A}_{12}\right)$$
$$= \det\left((s^2 + 2\alpha s + \alpha^2)I_N - \bar{A}_{21}\bar{A}_{12} - \alpha^2 I_N\right) = \det\left((s+\alpha)^2 I_N - \bar{A}_{21}\bar{A}_{12} - \alpha^2 I_N\right).$$

If we define the following the change of variable:

$$\mathcal{S} = (s + \alpha)^2 \tag{20}$$

then we can write

$$\det\left[\begin{array}{c|c} sI_N - \bar{A}_{11} & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & sI_N - \bar{A}_{22} \end{array}\right] = \det\left(\mathcal{S}I_N - (\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N)\right) \tag{21}$$

From this equation, the eigenvalues of $\bar{A}_{21}\bar{A}_{12}+\alpha^2 I_N$ that we denote by $\Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N\right)$, are the values of $\mathcal{S} = (s+\alpha)^2$ that also set $\det(sI_N - A)$ to zero. From (21), $\Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N\right) = (\Lambda(A) + \alpha)^2$, where $\Lambda(A)$ denotes the non zero eigenvalues of A.

From the eigenvalues properties,

$$\Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N\right) = \Lambda\left(\bar{A}_{21}\bar{A}_{12}\right) + \alpha^2. \tag{22}$$

The product $\bar{A}_{21}\bar{A}_{12}$ is

$$\bar{A}_{21}\bar{A}_{12} = \left(\frac{c}{\Delta\ell}\right)^2 \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}.$$

Using Fact 5.10.25 in [1, pp. 200]

$$\Lambda\left(\bar{A}_{21}\bar{A}_{12}\right) = -2\left(\frac{c}{\Delta\ell}\right)^2 \left(1 - \cos\left(\frac{k\pi}{N+1}\right)\right),\ k = 1,\ldots,N \tag{23}$$

Then

$$\Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N\right) = -2\left(\frac{c}{\Delta\ell}\right)^2 \left(1 - \cos\left(\frac{k\pi}{N+1}\right)\right) + \alpha^2,\ k = 1,\ldots,N \tag{24}$$

are the values of $\mathcal{S} = (s + \alpha)^2$ that set the characteristic equation of A to zero.

Consequently

$$(\Lambda(A) + \alpha)^2 = \Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2 I_N\right)$$

and from (20)–(24) the eigenvalues of A are $\Lambda(A) = -\alpha \pm \sqrt{-2\left(\frac{c}{\Delta\ell}\right)^2 \left(1 - \cos\left(\frac{k\pi}{N+1}\right)\right) + \alpha^2}$

That is

$$\Lambda(A) = -\alpha \pm j\sqrt{2\left(\frac{c}{\Delta\ell}\right)^2 \left(1 - \cos\left(\frac{k\pi}{N+1}\right)\right) - \alpha^2}$$

$$= -\alpha \pm j\sqrt{4\left(\frac{c}{\Delta\ell}\right)^2 \sin^2\left(\frac{k\pi}{2(N+1)}\right) - \alpha^2}$$

Recalling the definition of α in equation (3), we have:

$$\Lambda(A) = -\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m} \mp j\sqrt{\left(2\frac{c}{\Delta\ell}\sin\left(\frac{k\pi}{2(N+1)}\right)\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m}\right)^2},\ k = 1,\ldots,N$$

and this completes the proof.

$\square$

The asymptotic case of the nonzero eigenvalues is reported in the following corollary

Corollary 1 *If $N \to \infty$ then the eigenvalues of A are*

$$\lambda_0 = 0 \tag{25}$$

$$\lambda_{\pm k} = \underbrace{-\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m}}_{:=\alpha} \mp j\underbrace{\sqrt{\left(\frac{k\pi}{T_d}\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m}\right)^2}}_{:=b_k},\ k = 1, 2, \ldots \tag{26}$$

where L is the pipe length and T_d the time that a mass pressure takes to cross the pipeline, between its boundaries, at a constant speed c.

Proof: $\lim\limits_{N\to\infty} \lambda_0 = 0$ is trivial
Since $\Delta\ell$ is given by

$$\Delta\ell = \frac{L}{N}$$

then

$$\frac{c}{\Delta\ell} \sin\left(\frac{k\pi}{2(N+1)}\right) = \frac{cN}{L} \sin\left(\frac{k\pi}{2(N+1)}\right).$$

Taking the limit when $N \to \infty$, $\lim\limits_{N\to\infty} \frac{cN}{L} \sin\left(\frac{k\pi}{2(N+1)}\right) = \frac{c}{2L} k\pi$

and, consequently,

$$\lim_{N\to\infty} \lambda_{\pm k} = -\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m} \mp j\sqrt{\left(2\frac{c}{2L}k\pi\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m}\right)^2}$$

$$= -\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m} \mp j\sqrt{\left(\frac{c}{L}k\pi\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m}\right)^2}, \quad k = 1, 2, \dots$$

Given that $T_d = \dfrac{L}{c}$, one obtains

$$\lim_{N\to\infty} \lambda_{\mp k} = -\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m} \mp j\sqrt{\left(\frac{k\pi}{T_d}\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}AP_m}\right)^2}, \quad k = 1, 2, \dots$$

and this completes the proof.

$\square$

The zero eigenvalue means that there is an integrator in the pipeline model.

It has associated an eigenvector v_0, which defines a direction in the state-space where the pipeline behaves like a pure integrator. The following lemma gives the value of this eigenvector:

Lemma 1 *Consider A as in (12)–(15). Then*

$$v_0 = e_1 + e_2 + \cdots + e_{N+1} \tag{27}$$

where e_i is the ith vector of the canonical orthonormal base in $\mathbb{R}^{2N+1}$, is the eigenvector associated to the zero eigenvalue, i. e., $Av_0 = 0$.

Proof: Let us denote v_0 as

$$v_0 = \begin{bmatrix} v \\ 0_N \end{bmatrix} \tag{28}$$

where 0_N is the zero vector in R^N and

$$v = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{N+1} \in \mathbb{R}^{N+1} \tag{29}$$

where ε_i is the ith vector of the canonical orthonormal basis in R^{N+1}. Using (11) and (28)

$$Av_0 = \begin{bmatrix} A_{11}v \\ A_{21}v \end{bmatrix} \tag{30}$$

Given that $A_{11} = 0_{(N+1)\times(N+1)}$ then $A_{11}v = 0$. From (14) and (29)

$$A_{21}v = \begin{bmatrix} \frac{\mathcal{A}}{\Delta\ell} - \frac{\mathcal{A}}{\Delta\ell} \\ \frac{\mathcal{A}}{\Delta\ell} - \frac{\mathcal{A}}{\Delta\ell} \\ \vdots \\ \frac{\mathcal{A}}{\Delta\ell} - \frac{\mathcal{A}}{\Delta\ell} \end{bmatrix} = 0_N$$

and this completes the proof.

□

Remark 1 *In Lemma 1, to prove the existence of the eigenvalue associated to the zero eigenvalue we only used the submatrices A_{11} and A_{12}. Therefore, we can say that this eigenvalue is generated by these submatrices. The nonlinearity of the model is only expressed by matrix A_{22}. For this reason, if we decompose the full nonlinear model into a linear subsystem in cascade with a nonlinear one, the zero eigenvalue would appear in the linear subsystem indicating the presence of an integrator in the full model. Also, A_{11} and A_{12} depend neither on p_m nor on q_m.*

4   Transfer functions characterisation

We determine the transfer function. Recall that the massflow at the boundaries were chosen to be our inputs and the pressure at the boundaries our outputs.

To start, we apply the Laplace transform to (10) and obtain:

$$Y(s) = C(sI - A)^{-1} BU(s)$$

where $Y(s) = \begin{bmatrix} Y_1(s) & Y_2(s) \end{bmatrix}^T$ and $U(s) = \begin{bmatrix} U_1(s) & U_2(s) \end{bmatrix}^T$, and $F(s)$ denotes the Laplace transform of $f(t)$. That is

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$

Also:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} C_1 (sI - A)^{-1} B_1 & C_1 (sI - A)^{-1} B_2 \\ C_2 (sI - A)^{-1} B_1 & C_2 (sI - A)^{-1} B_2 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \tag{31}$$

and also

$$
\begin{aligned}
G_{11}(s) &= = \left.\frac{Y_1(s)}{U_1(s)}\right|_{U_2(s)=0} = \left.\frac{P_0(s)}{Q_0(s)}\right|_{Q_{N+1}(s)=0} \\[2mm]
G_{22}(s) &= \left.\frac{Y_2(s)}{U_2(s)}\right|_{U_1(s)=0} = \left.\frac{P_N(s)}{Q_{N+1}(s)}\right|_{Q_0(s)=0} \\[2mm]
G_{12}(s) &= \left.\frac{Y_1(s)}{U_2(s)}\right|_{U_1(s)=0} = \left.\frac{P_0(s)}{Q_{N+1}(s)}\right|_{Q_0(s)=0} \\[2mm]
G_{21}(s) &= \left.\frac{Y_2(s)}{U_1(s)}\right|_{U_2(s)=0} = \left.\frac{P_N(s)}{Q_0(s)}\right|_{Q_{N+1}(s)=0}.
\end{aligned}
\tag{32}
$$

4.1 Transfer function G_{11}

When we select this transfer function, it means that we are interested in the transfer function between the pressure and massflow at the intake node, i.e. $\dfrac{Y_1(s)}{U_1(s)}$ when $U_2(s) = 0$ that is:

$$
G_{11}(s) = \left.\frac{P_0(s)}{Q_0(s)}\right|_{Q_1(s)=0} = \left.\frac{Y_1(s)}{U_1(s)}\right|_{U_2(s)=0}
$$

and hence:

$$
G_{11}(s) = C_1 \left(sI - A\right)^{-1} B_1,
\tag{33}
$$

where B_1 and C_1 are the first column and first row of B and C in (16)–(17), respectively. $G_{11}(s)$ is a rational function whose poles are the eigenvalues of A.

The following theorem states the zeros of this transfer function.

Theorem 2 *The zeros of $G_{11}(s)$ are*

$$
z_{\pm k} = -\frac{f_c c^2 Q_m}{4\mathcal{D}A P_m} \pm j\sqrt{\left(2\frac{c}{\Delta\ell}\sin\left(\frac{(2k-1)\pi}{2(2N+1)}\right)\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}A P_m}\right)^2}, \quad k = 1, \ldots, N
\tag{34}
$$

Proof: To do this we recall the result from [2, pp. 284], we have that the zeros of the transfer function (33) are the zeros of the following polynomial

$$
\left|\begin{array}{c|c} sI_{(2N+1)} - A & -B_1 \\ \hline C_1 & 0 \end{array}\right| = 0.
\tag{35}
$$

Recalling that $C_1 = e_1 \in \mathbb{R}^{(2N+1)}$ and $B_1 = \dfrac{c^2}{\mathcal{A}\Delta\ell}e_1\mathbb{R}^{(2N+1)}$ (see equations (16)–(17)), we have:

$$\left| \begin{array}{c|c} sI_{(2N+1)} - A & -e_1 \\ \hline e_1^T & 0 \end{array} \right| = 0 \quad \Leftrightarrow \tag{36}$$

$$\Leftrightarrow \left| \begin{array}{c|c} \left[\begin{array}{c|c} sI_{N+1} - A_{11} & -A_{12} \\ \hline -A_{21} & sI_N - A_{22} \end{array} \right] & \begin{array}{c} -1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline \begin{array}{ccccccc} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{array} & 0 \end{array} \right| = 0 \quad \Leftrightarrow \tag{37}$$

using the definition of A_{11} and A_{22} in (12) and (15)

$$\Leftrightarrow \left| \begin{array}{c|c} \left[\begin{array}{c|c} sI_{N+1} & -A_{12} \\ \hline -A_{21} & (s+2\alpha)I_N \end{array} \right] & \begin{array}{c} -1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline \begin{array}{cccccc} 1 & 0 & \cdots & \cdots & \cdots & 0 \end{array} & 0 \end{array} \right| = 0. \tag{38}$$

We develop this determinant first along the last column and next along the last row, and obtain:

$$\left| \begin{array}{c|c} sI_N & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & (s+2\alpha)I_N \end{array} \right| = 0 \tag{39}$$

where $\bar{A}_{12}$ denotes matrix A_{12} without the 1st row and $\bar{A}_{21}$ denotes matrix A_{21} without the 1st column. Next, as $sI_N\left(-\bar{A}_{12}\right) = \left(-\bar{A}_{12}\right)sI_N$, we apply again Fact 2.13.10 in [1, pp. 62–63], which states that for the arbitrary matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D} \in \mathbb{R}^{N\times N}$ such that $\mathbf{AB} = \mathbf{BA}$ then

$$\det\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right] = \det(\mathbf{DA} - \mathbf{CB})$$

and

$$\left| \begin{array}{c|c} sI_N & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & (s+2\alpha)I_N \end{array} \right| = \left| s(s+2\alpha)I_N - \bar{A}_{21}\bar{A}_{12} \right| \tag{40}$$

$$= \left| \left(s^2 + 2\alpha s + \alpha^2\right)I_N - \left(\bar{A}_{21}\bar{A}_{12} + \alpha^2\right) \right| \tag{41}$$

$$= \left| (s+\alpha)^2 I_N - \left(\bar{A}_{21}\bar{A}_{12} + \alpha^2\right) \right| \tag{42}$$

We do the same change of variable as before

$$\mathcal{S} = (s+\alpha)^2 \tag{43}$$

and then can write:

$$\left| \begin{array}{c|c} sI_N & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & (s+2\alpha)I_N \end{array} \right| = \left| \mathcal{S}I_N - \left(\bar{A}_{21}\bar{A}_{12} + \alpha^2\right) \right| \tag{44}$$

Next, we calculate the spectrum of matrix $\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2\right)$, that is:

$$\Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2\right) = \Lambda\left(\bar{A}_{21}\bar{A}_{12}\right) + \alpha^2$$

Now, we calculate the product:

$$\bar{A}_{21}\bar{A}_{12} = \begin{bmatrix} -\dfrac{\mathcal{A}}{\Delta\ell} & 0 & \cdots & 0 & 0 \\ \dfrac{\mathcal{A}}{\Delta\ell} & -\dfrac{\mathcal{A}}{\Delta\ell} & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \cdots & \cdots \\ \vdots & \cdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & \dfrac{\mathcal{A}}{\Delta\ell} & -\dfrac{\mathcal{A}}{\Delta\ell} \end{bmatrix} \begin{bmatrix} \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} \end{bmatrix}$$

$$= \left(\frac{c}{\Delta\ell}\right)^2 \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \in \mathbb{R}^N$$

Then

$$\Lambda\left(\bar{A}_{21}\bar{A}_{12}\right) = \left(\frac{c}{\Delta\ell}\right)^2 \Lambda\left(M_N\right)$$

where

$$M_N = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \in \mathbb{R}^{N\times N}. \tag{45}$$

From [4, pp. 72]

$$\Lambda\left(M_N\right) = -2 + 2\cos\left(\frac{(2k-1)\pi}{2N+1}\right), \quad k = 1, 2, 3, \ldots, N. \tag{46}$$

Having that

$$\Lambda\left(\bar{A}_{21}\bar{A}_{12} + \alpha^2\right) = -2\left(\frac{c}{\Delta\ell}\right)^2 \left(1 - \cos\left(\frac{(2k-1)\pi}{2N+1}\right)\right) + \alpha^2 \tag{47}$$

$$= j^2 \left(2\left(\frac{c}{\Delta\ell}\right)^2 \left(1 - \cos\left(\frac{(2k-1)\pi}{2N+1}\right)\right) - \alpha^2\right) \tag{48}$$

then from (43):

$$
\begin{aligned}
z_{\pm k} &= -\alpha \pm j\sqrt{2\left(\frac{c}{\Delta\ell}\right)^2\left(1 - \cos\left(\frac{(2k-1)\pi}{2N+1}\right)\right) - \alpha^2} \\
&= -\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m} \pm j\sqrt{4\left(\frac{c}{\Delta\ell}\right)^2\sin^2\left(\frac{(2k-1)\pi}{2(2N+1)}\right) - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2} \\
&= -\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m} \pm j\sqrt{\left(2\frac{c}{\Delta\ell}\sin\left(\frac{(2k-1)\pi}{2(2N+1)}\right)\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2}, \quad k = 1,\ldots,N
\end{aligned}
$$

and this completes the proof.

□

The following corollary resolves the asymptotic case.

Corollary 2 *If $N \to \infty$ then the zero of $G_{11}(s)$ are*

$$
z_{\pm k} = -\underbrace{\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}}_{:=\alpha} \pm j\underbrace{\sqrt{\left(\frac{(2k-1)\pi}{2T_d}\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2}}_{:=\beta_k}, \quad k = 1, 2, \ldots \tag{49}
$$

where L is the pipe length and T_d the time that a particle of gas takes to cross the pipeline between its boundaries, at a constant speed c.

Proof: Since $\Delta\ell$ is given by

$$
\Delta\ell = \frac{L}{N}
$$

then

$$
\frac{c}{\Delta\ell}\sin\left(\frac{(2k-1)\pi}{2(2N+1)}\right) = \frac{cN}{L}\sin\left(\frac{(2k-1)\pi}{2(2N+1)}\right).
$$

Taking the limit when $N \to \infty$,

$$
\lim_{N\to\infty}\frac{cN}{L}\sin\left(\frac{(2k-1)\pi}{2(2N+1)}\right) = \lim_{N\to\infty}\frac{cN(2k-1)\pi}{2(2N+1)L}\lim_{N\to\infty}\frac{\sin\left(\frac{(2k-1)\pi}{2(2N+1)}\right)}{\frac{(2k-1)\pi}{2(2N+1)}} = \frac{c(2k-1)\pi}{4L}
$$

and, consequently,

$$
\lim_{N\to\infty}z_{\pm k} = -\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m} \pm j\sqrt{\left(\frac{c(2k-1)\pi}{2L}\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2} \quad k = 1, 2, \ldots
$$

Given that $T_d = \dfrac{L}{c}$ then

$$
\lim_{N\to\infty}z_{\pm k} = -\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m} \pm j\sqrt{\left(\frac{(2k-1)\pi}{2Td}\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2} \quad k = 1, 2, \ldots
$$

and this completes the proof.

□

Corollary 3 *Transfer function G_{11} has the following form:*

$$G_{11} = \frac{K_G \prod\limits_{k=1}^{\infty} \left(\dfrac{s^2}{z_k z_{-k}} + s \left(\dfrac{1}{z_k} + \dfrac{1}{z_{-k}} \right) + 1 \right)}{s \prod\limits_{k=1}^{\infty} \left(\dfrac{s^2}{\lambda_k \lambda_{-k}} + s \left(\dfrac{1}{\lambda_k} + \dfrac{1}{\lambda_{-k}} \right) + 1 \right)} \tag{50}$$

where $z_k = \alpha + j\beta_k$ and $z_{-k} = \alpha - j\beta_k$, as well as $\lambda_k = \alpha + jb_k$ and $\lambda_{-k} = \alpha - jb_k$, as defined in Corollary 2 and λ_i are defined in Corollary 1.

Proof: From Corollary 2 and Corollary 1, we can write:

$$G_{11} = \frac{K_G \prod\limits_{k=1}^{\infty} \left(\dfrac{s}{-z_k} + 1 \right) e^{\frac{s}{\beta_k}} \left(\dfrac{s}{-z_{-k}} + 1 \right) e^{\frac{s}{-\beta_k}}}{s \prod\limits_{k=1}^{\infty} \left(\dfrac{s}{-\lambda_k} + 1 \right) e^{\frac{s}{b_k}} \left(\dfrac{s}{-\lambda_{-k}} + 1 \right) e^{\frac{s}{-b_k}}} \tag{51}$$

and expression (50) follows immediately, after calculating the products:

$$\left(\frac{s}{-z_k} + 1 \right) \left(\frac{s}{-z_{-k}} + 1 \right) \text{ and } \left(\frac{s}{-\lambda_k} + 1 \right) \left(\frac{s}{-\lambda_{-k}} + 1 \right).$$

$\square$

To complete the transfer function characterisation we need to compute the gain K_G.

Theorem 3 *Consider B_1, the first column of B defined in (16), written in the base*

$$\{v_{-N}, \ldots, v_{-1}, v_0, v_1, \ldots, v_N\}$$

where v_i, $i = -N, \ldots, -1, 0, 1, \ldots, N$, are the eigenvectores of A. If ϑ_0 is the component of B_1 along v_0 then

$$K_G = \vartheta_0.$$

Proof: Denote the zero eigenvalue of A as λ_0 and λ_i, $i = -N, \cdots - 1, 1, \ldots, N$ the complex eigenvalues, i.e. $\lambda_{-i} = \lambda_i^*$ where $\cdot^*$ means the conjugate eigenvalue. Since all eigenvalues have multiplicity one there are 2N+1 independent eigenvectors v_i, $i = -N, \ldots, N$, respectively associated to each eigenvalue λ_i. Thus v_i and $(s - \lambda_i)$, $i = -N, \ldots, -1, 0, 1, \ldots, N$ are, respectively, the eigenvectors and the eigenvalues of $sI - A$. Consequently,

$$(SI - A)^{-1} v_i = \frac{1}{s - \lambda_i} v_i \tag{52}$$

i.e., $\dfrac{1}{s - \lambda_i}$, $i = -N, \ldots, N$ are eigenvalues of $(sI - A)^{-1}$ associated to the eigenvectors v_i. Then we

can write:

$$\Lambda = \begin{bmatrix} \dfrac{1}{s - \lambda_{-N}} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \dfrac{1}{s - \lambda_{-N+1}} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ \vdots & \vdots & \cdots & \dfrac{1}{s - \lambda_0} & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots & \dfrac{1}{s - \lambda_N} \end{bmatrix} \tag{53}$$

Also define the similarity matrix T, considering the $2N + 1$ independent eigenvectors v_i:

$$T = \begin{bmatrix} v_{-N} & \cdots & v_{-1} & v_0 & v_1 & \cdots & v_N \end{bmatrix} \tag{54}$$

And we can write:

$$(SI - A)^{-1} = T\Lambda T^{-1}. \tag{55}$$

If we decompose B_1 into directions v_i, i. e.,

$$B_1 = \vartheta_{-N} v_{-N} + \cdots + \vartheta_{-1} v_{-1} + \vartheta_0 v_0 + \vartheta_1 v_1 + \cdots + \vartheta_N v_N = T \begin{bmatrix} \vartheta_{-N} \\ \vdots \\ \vartheta_N \end{bmatrix} \tag{56}$$

then we can express the transfer function $G_{11}(s)$ as

$$\begin{aligned} G_{11}(s) &= C_1 (sI - A)^{-1} B_1 = C_1 T\Lambda T^{-1} T \begin{bmatrix} \vartheta_{-N} \\ \vdots \\ \vartheta_N \end{bmatrix} = C_1 T\Lambda \begin{bmatrix} \vartheta_{-N} \\ \vdots \\ \vartheta_N \end{bmatrix} = \\ &= C_1 \left(\frac{\vartheta_{-N}}{s - \lambda_{-N}} v_{-N} + \cdots \frac{\vartheta_{-1}}{s - \lambda_{-1}} v_{-1} + \frac{\vartheta_0}{s} v_0 + \frac{\vartheta_1}{s - \lambda_1} v_1 + \cdots + \frac{\vartheta_N}{s - \lambda_N} v_N \right). \end{aligned} \tag{57}$$

After multiplying (57) by s one obtains:

$$\begin{aligned} K_G &= C_1 \lim_{s \to 0} \left(\frac{\vartheta_{-N} s}{s - \lambda_{-N}} v_{-N} + \cdots + \frac{\vartheta_{-1} s}{s - \lambda_{-1}} v_{-1} + \frac{\vartheta_0 s}{s} v_0 + \frac{\vartheta_1 s}{s - \lambda_1} v_1 + \cdots + \frac{\vartheta_N s}{s - \lambda_N} v_N \right) \\ &= \vartheta_0 C_1 v_0 = \vartheta_0, \end{aligned}$$

$\square$

because $C_1 = e_1^T$ and $v_0 = e_1 + e_2 + \cdots + e_{N+1}$. If we knew all the eigenvectors we could straightforwardly determine ϑ_0. But, only v_0 is known and it is not so immediate to compute ϑ_0. The next lemma is of good help to solve this problem.

Lemma 2 : *If $A \in \mathbb{R}^{n \times n}$ is a singular matrix, v_0 its eigenvector associated to the zero eigenvalue and $A^T v_0 = 0$, then v_0 is orthogonal to the remaining eigenvectors v_i, $i \neq 0$, of A.*

Proof: Let v_i with $i \neq 0$ the eigenvector of A associated to the eigenvalue $\lambda_i \neq 0$. By the eigenvector definition

$$A v_i = \lambda_i v_i \Rightarrow v_i = \frac{1}{\lambda_i} A v_i. \tag{58}$$

Now, using this equation, we compute the internal product between v_i and v_0,

$$\langle v_i, v_0 \rangle = v_i^T v_0 = \left(\frac{1}{\lambda_i} A v_i \right)^T v_0 = \frac{1}{\lambda_i} v_i^T A^T v_0 = 0 \tag{59}$$

from which conclude that v_0 and v_i are orthogonal.

$\square$

Corollary 4 *Consider A as defined in (12)–(15). Its eigenvectors v_i, $i = -N, \ldots, -1, 1, \ldots, N$, are orthogonal to v_0.*

Proof: Recalling the definitions of v_0 and A_{11} in equations (28) and (12), respectively, then:

$$A^T v_0 = \left[\begin{array}{c|c} A_{11}^T & A_{21}^T \\ \hline A_{12}^T & A_{22}^T \end{array} \right] v_0 = \left[\begin{array}{c} 0_{(N+1) \times (N+1)} \\ A_{12}^T v \end{array} \right]$$

Computing

$$A_{12}^T v = \begin{bmatrix} -\dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 & 0 & \cdots & 0 & 0 \\ \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \vdots & \cdots & \vdots & \vdots \\ 0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -\dfrac{c^2}{\mathcal{A}\Delta\ell} \\ 0 & 0 & 0 & \cdots & 0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 & \cdots & 0 & 0 \\ 0 & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \dfrac{c^2}{\mathcal{A}\Delta\ell} & \cdots & \vdots & \vdots \\ \vdots & 0 & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 \\ 0 & 0 & 0 & \cdots & -\dfrac{c^2}{\mathcal{A}\Delta\ell} & \dfrac{c^2}{\mathcal{A}\Delta\ell} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{c^2}{\mathcal{A}\Delta\ell} - \dfrac{c^2}{\mathcal{A}\Delta\ell} \\ \vdots \\ \dfrac{c^2}{\mathcal{A}\Delta\ell} - \dfrac{c^2}{\mathcal{A}\Delta\ell} \end{bmatrix} = 0,$$

Then by Lemma 2 we have the expected result.

□

Corollary 5 $K_G = \dfrac{c^2}{\mathcal{A}L}$.

Proof: Decompose B_1 as

$$B_1 = P_{v_0} B_1 + P_{v_0^{\perp}} B_1 \tag{60}$$

where P_w is the orthogonal projection into w operator and $w^{\perp}$ denotes the orthogonal complement of w. From the orthogonality condition between v_0 and v_i, $i = -N, \ldots, -1, 1, \ldots, N$,

$$P_{v_0} B_1 = \vartheta_0 v_0. \tag{61}$$

Given that

$$P_{v_0} B_1 = (v_0^T v_0)^{-1} v_0^T B_1 v_0, \tag{62}$$

then

$$\vartheta_0 = (v_0^T v_0)^{-1} v_0^T B_1 = \frac{c^2}{\mathcal{A}(N+1)\Delta \ell}. \tag{63}$$

Replacing $\Delta \ell$ by $\dfrac{L}{N}$ we find

$$\vartheta_0 = \frac{c^2 N}{\mathcal{A}(N+1)L}. \tag{64}$$

When $N \to \infty \Rightarrow \dfrac{N}{N+1} \to 1$,

$$\vartheta_0 = \frac{c^2}{\mathcal{A}L}. \tag{65}$$

and $K_G = \vartheta_0 = \dfrac{c^2}{\mathcal{A}L}$.

□

Corollary 6 *Transfer function G_{11} has the following form, according to Corollary 3:*

$$G_{11} = \frac{c^2}{\mathcal{A}L} \frac{\prod_{k=1}^{\infty} \left(\dfrac{s^2}{z_k z_{-k}} + s \left(\dfrac{1}{z_k} + \dfrac{1}{z_{-k}} \right) + 1 \right)}{s \prod_{k=1}^{\infty} \left(\dfrac{s^2}{\lambda_k \lambda_{-k}} + s \left(\dfrac{1}{\lambda_k} + \dfrac{1}{\lambda_{-k}} \right) + 1 \right)} \tag{66}$$

where z_k are defined in Corollary 2 and λ_k are defined in Corollary 1.

4.2 Transfer function G_{22}

Next, we determine the transfer function

$$G_{22} = \frac{Y_2(s)}{U_2(s)} = \frac{P_N(s)}{Q_N(s)} \tag{67}$$

when $U_1(s) = 0$. That is the ratio between the pressure and the massflow at the offtake node. From (31):

$$G_{22}(s) = C_2 \left(sI - A\right)^{-1} B_2, \tag{68}$$

where C_2 is the second column of C and B_2 is the second row of B.

Similarly to what happens with $G_{11}(s)$, $G_{22}(s)$ is a rational function whose poles are the eigenvalues of A. Following the same methodology as for $G_{11}(s)$, we would like to calculate the zeros of $G_{22}(s)$ in order to investigate pole-zero cancelations.

Theorem 4 *The zeros of $G_{22}(s)$ are*

$$z_k = -\frac{f_c c^2 Q_m}{4 \mathcal{D} \mathcal{A} P_m} \pm j \sqrt{\left(2\frac{c}{\Delta\ell}\sin\left(\frac{(2k-1)\pi}{2N+1}\right)\right)^2 - \left(\frac{f_c c^2 Q_m}{4\mathcal{D}\mathcal{A}P_m}\right)^2}, \quad k = 1,\ldots,N \tag{69}$$

Proof: The proof is very similar to the one of Theorem 2. Again, we recall the result from [2, pp. 284] that states that the zeros of the transfer function (67) are the zeros of the following polynomial:

$$\left| \begin{array}{c|c} sI_{(2N+1)} - A & -B_2 \\ \hline C_2 & 0 \end{array} \right| = 0 \tag{70}$$

Recall the definitions of $C_2 = e_{N+1} \in \mathbb{R}^{(2N+1)}$ and $B_2 = -\dfrac{c^2}{\mathcal{A}\Delta\ell} e_{N+1} \mathbb{R}^{(2N+1)}$ and the proof follows exactly as for Theorem 2.

We have:

$$\left| \begin{array}{c|c} sI_{(2N+1)} - A & e_{N+1} \\ \hline e_{N+1}^T & 0 \end{array} \right| = 0 \quad \Leftrightarrow \tag{71}$$

$$\Leftrightarrow \left| \begin{array}{c|c} \left[\begin{array}{c|c} sI_{N+1} - A_{11} & -A_{12} \\ \hline -A_{21} & sI_N - A_{22} \end{array} \right] & \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{array} \\ \hline 0 \cdots \cdots 0 \;\Big|\; 1 \cdots \cdots 0 & 0 \end{array} \right| \tag{72}$$

From the definition of A_{11} and A_{22} defined in (12) and (15)

$$= 0 \quad \Leftrightarrow \tag{73}$$

$$\Leftrightarrow \left| \begin{array}{c|c} \left[\begin{array}{c|c} sI_{N+1} & -A_{12} \\ \hline -A_{21} & (s+2\alpha)I_N \end{array} \right] & \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{array} \\ \hline 0 \cdots 0 \;\Big|\; 1 \cdots 0 & 0 \end{array} \right| = 0. \tag{74}$$

We develop this determinant first along the last column and next along the last row, and obtain:

$$
\begin{vmatrix} sI_N & -\bar{A}_{12} \\ \hline -\bar{A}_{21} & (s+2\alpha)I_N \end{vmatrix} = 0 \tag{75}
$$

where $\bar{A}_{12}$ denotes matrix A_{12} without the 1st column and $\bar{A}_{21}$ denotes matrix A_{21} without the 1st row. Next, as $sI_N\left(-\bar{A}_{12}\right) = \left(-\bar{A}_{12}\right)sI_N$, we apply again Fact 2.13.10 in [1, pp. 62–63], that states that for the arbitrary matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D} \in \mathbb{R}^{N\times N}$ such that $\mathbf{AB}=\mathbf{BA}$ then

$$
\det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{DA}-\mathbf{CB})
$$

and

$$
\begin{vmatrix} sI_N & -\bar{A}_{12} \\ -\bar{A}_{21} & (s+2\alpha)I_N \end{vmatrix} = \left| s(s+2\alpha)I_N - \bar{A}_{21}\bar{A}_{12} \right| \tag{76}
$$

$$
= \left| \left(s^2+2\alpha s+\alpha^2\right)I_N - \left(\bar{A}_{21}\bar{A}_{12}+\alpha^2\right) \right| \tag{77}
$$

$$
= \left| (s+\alpha)^2 I_N - \left(\bar{A}_{21}\bar{A}_{12}+\alpha^2\right) \right|. \tag{78}
$$

We do the usual change of variable

$$
\mathcal{S} = (s+\alpha)^2 \tag{79}
$$

and then can write:

$$
= \left| \mathcal{S}I_N - \left(\bar{A}_{21}\bar{A}_{12}+\alpha^2\right) \right| \tag{80}
$$

Next, we calculate the spectrum of matrix $\left(\bar{A}_{21}\bar{A}_{12}+\alpha^2\right)$, that is:

$$
\Lambda\left(\bar{A}_{21}\bar{A}_{12}+\alpha^2\right) = \Lambda\left(\bar{A}_{21}\bar{A}_{12}\right)+\alpha^2
$$

Now, we calculate the product:

$$
\bar{A}_{21}\bar{A}_{12} =
\begin{bmatrix}
-\frac{\mathcal{A}}{\Delta\ell} & 0 & \cdots & 0 & 0 \\
\frac{\mathcal{A}}{\Delta\ell} & -\frac{\mathcal{A}}{\Delta\ell} & \cdots & \vdots & \vdots \\
0 & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{\mathcal{A}}{\Delta\ell} & -\frac{\mathcal{A}}{\Delta\ell} \\
0 & 0 & \cdots & 0 & \frac{\mathcal{A}}{\Delta\ell}
\end{bmatrix}
\begin{bmatrix}
\frac{c^2}{\mathcal{A}\Delta\ell} & -\frac{c^2}{\mathcal{A}\Delta\ell} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{c^2}{\mathcal{A}\Delta\ell} & -\frac{c^2}{\mathcal{A}\Delta\ell} & 0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & -\frac{c^2}{\mathcal{A}\Delta\ell} & 0 \\
0 & 0 & 0 & 0 & \cdots & \frac{c^2}{\mathcal{A}\Delta\ell} & -\frac{c^2}{\mathcal{A}\Delta\ell}
\end{bmatrix}
$$

$$
= \left(\frac{c}{\Delta\ell}\right)^2
\begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 & 0 \\
1 & -2 & 1 & \cdots & 0 & 0 \\
0 & 1 & -2 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & -2 & 1 \\
0 & 0 & 0 & \cdots & 1 & -1
\end{bmatrix} \in \mathbb{R}^N
$$

Then

$$\left| sI_N - \bar{A}_{21}\bar{A}_{12} \right| = \left(\frac{c}{\Delta \ell} \right)^2 \Lambda \left(M_N \right)$$

where

$$M_N = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \in \mathbb{R}^{N \times N}. \tag{81}$$

From [4, pp. 72]

$$\Lambda \left(M_N \right) = -2 + 2 \cos \left(\frac{(2k-1)\,\pi}{2N+1} \right), \quad k = 1, 2, 3, \ldots, N. \tag{82}$$

and from here the proof follows exactly as for the calculus of the zeros of $G_{11}(s)$, and we can see that are the same.

Likewise follows for the asymptotic case. As we can see from the definition of the transfer function (33) and (68) as well as from the definition of the $B_i, C_i, i = 1, 2$ we have

$$G_{11}(s) = -G_{22}(s) \tag{83}$$

Therefore, its zeros will be necessarily coincident.

$\square$

4.3   Transfer function G_{12}

Acccording to (31), consider now the transfer function

$$G_{12}(s) = \frac{Y_1(s)}{U_2(s)} = \frac{P_0(s)}{Q_{N+1}(s)} \tag{84}$$

with $U_1(s) = Q_0(s) = 0$, or equivalently:

$$G_{12}(s) \;=\; C_1 \left(sI - A \right)^{-1} B_2, \tag{85}$$

where C_1 is the first column of C and B_2 is the second row of B.

Theorem 5 *The transfer function $G_{12}(s)$ has no zeros.*

Proof: According to [2, pp. 284], we have that the zeros of the transfer function (84) are the zeros of the following polynomial:

$$\left| \begin{array}{c|c} sI_{(2N+1)} - A & -B_2 \\ \hline C_1 & 0 \end{array} \right|$$

and recalling that $C_1 = e_1 \in \mathbb{R}^{(2N+1)}$ and $B_2 = -e_{N+1}\mathbb{R}^{(2N+1)}$, we have:

$$\left| \begin{array}{c|c} sI_{(2N+1)} - A & e_{N+1} \\ \hline e_1^T & 0 \end{array} \right| = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \left| \begin{array}{c} \left[\begin{array}{c|c} sI_{N+1} - A_{11} & -A_{12} \\ \hline -A_{21} & sI_N - A_{22} \end{array} \right] \quad \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline 1 \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \quad \Big| \quad 0 \end{array} \right|$$

(86)

And from de definition of A_{11} and A_{22} in (12) and (15)

$$= 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \left| \begin{array}{c} \left[\begin{array}{c|c} sI_{N+1} & -A_{12} \\ \hline -A_{21} & (s-\alpha)I_N \end{array} \right] \quad \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline 1 \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad 0 \quad \Big| \quad 0 \end{array} \right| = 0$$

(87)

We develop this determinant first along the last column and next along the last row, and obtain:

$$\left| \begin{array}{ccccccc|cccccc} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma & 0 & 0 & \cdots & 0 & 0 & 0 \\ s & 0 & 0 & \cdots & 0 & 0 & 0 & -\gamma & \gamma & 0 & \cdots & \vdots & \vdots & \vdots \\ 0 & s & 0 & \cdots & 0 & 0 & 0 & 0 & -\gamma & \gamma & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & s & 0 & 0 & 0 & 0 & 0 & \cdots & -\gamma & \gamma & 0 \\ 0 & 0 & 0 & \cdots & 0 & s & 0 & 0 & 0 & 0 & \cdots & 0 & -\gamma & \gamma \\ \hline \varrho & 0 & 0 & \cdots & 0 & 0 & 0 & (s+2\alpha) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\varrho & \varrho & 0 & \cdots & \vdots & \vdots & \vdots & 0 & (s+2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots \\ 0 & -\varrho & \varrho & \cdots & \vdots & \vdots & \vdots & 0 & 0 & (s+2\alpha) & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & -\varrho & \varrho & 0 & \vdots & \vdots & \vdots & \cdots & 0 & (s+2\alpha) & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\varrho & \varrho & 0 & 0 & 0 & \cdots & 0 & 0 & (s+2\alpha) \end{array} \right| = 0$$

(88)

$$\gamma = -\frac{c^2}{\mathcal{A}\Delta\ell} \text{ and } \varrho = -\frac{\mathcal{A}}{\Delta\ell}.$$

We don't worry about the signs of the cofactor, since our aim is to determine the zeros of the determinant. Now, we develop this determinant first along the first line, and we obtain:

$$
\gamma
\left|
\begin{array}{ccccccc|cccccc}
s & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & 0 \\
0 & s & 0 & \cdots & \vdots & \vdots & \vdots & -\gamma & \gamma & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & s & 0 & 0 & \vdots & \vdots & \cdots & -\gamma & \gamma & 0 \\
0 & 0 & 0 & \cdots & 0 & s & 0 & 0 & 0 & \cdots & 0 & -\gamma & \gamma \\ \hline
\varrho & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\varrho & \varrho & 0 & \cdots & \vdots & \vdots & 0 & (s+2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots \\
0 & -\varrho & \varrho & \cdots & \vdots & \vdots & 0 & 0 & (s+2\alpha) & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & -\varrho & \varrho & 0 & \vdots & \vdots & \cdots & \vdots & (s+2\alpha) & 0 \\
0 & 0 & 0 & \cdots & 0 & -\varrho & \varrho & 0 & 0 & \cdots & 0 & 0 & (s+2\alpha)
\end{array}
\right| = 0
$$

Next, we develop along column–N:

$$
\gamma\varrho
\left|
\begin{array}{cccccc|cccccc}
s & 0 & 0 & \cdots & 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & 0 \\
0 & s & 0 & \cdots & \vdots & \vdots & -\gamma & \gamma & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & s & 0 & \vdots & \vdots & \cdots & -\gamma & \gamma & 0 \\
0 & 0 & 0 & \cdots & 0 & s & 0 & 0 & \cdots & 0 & -\gamma & \gamma \\ \hline
\varrho & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\varrho & \varrho & 0 & \cdots & \vdots & \vdots & (s+2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots \\
0 & -\varrho & \varrho & \cdots & 0 & 0 & 0 & (s+2\alpha) & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\varrho & \varrho & 0 & 0 & \cdots & 0 & (s+2\alpha) & 0
\end{array}
\right| = 0
$$

Here the matrix is of dimension $2(N-1) \times 2(N-1)$.

Swap the first row of blocks with the second one:

$$
\gamma\varrho
\left|
\begin{array}{cccccc|cccccc}
\varrho & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\varrho & \varrho & 0 & \cdots & \vdots & \vdots & (s+2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots \\
0 & -\varrho & \varrho & \cdots & \vdots & \vdots & 0 & (s+2\alpha) & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\varrho & \varrho & 0 & 0 & \cdots & 0 & (s+2\alpha) & 0 \\ \hline
s & 0 & 0 & \cdots & 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & 0 \\
0 & s & 0 & \cdots & \vdots & \vdots & -\gamma & \gamma & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & s & 0 & \vdots & \vdots & \cdots & -\gamma & \gamma & 0 \\
0 & 0 & 0 & \cdots & 0 & s & 0 & 0 & \cdots & 0 & -\gamma & \gamma
\end{array}
\right| = 0
$$

Swap the first column of blocks with the second one:

$$
\gamma\varrho
\left|
\begin{array}{ccccccc|ccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \varrho & 0 & 0 & \cdots & 0 & 0 \\
(s+2\alpha) & 0 & \cdots & \vdots & \vdots & \vdots & -\varrho & \varrho & 0 & \cdots & \vdots & \vdots \\
0 & (s+2\alpha) & \cdots & \vdots & \vdots & \vdots & 0 & -\varrho & \varrho & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & (s+2\alpha) & 0 & 0 & 0 & 0 & \cdots & -\varrho & \varrho \\
\gamma & 0 & \cdots & 0 & 0 & 0 & s & 0 & 0 & \cdots & 0 & 0 \\
-\gamma & \gamma & \cdots & \vdots & \vdots & \vdots & 0 & s & 0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & -\gamma & \gamma & 0 & \vdots & \vdots & \vdots & \cdots & s & 0 \\
0 & 0 & \cdots & 0 & -\gamma & \gamma & 0 & 0 & 0 & \cdots & 0 & s
\end{array}
\right| = 0
$$

Again develop the determinant along the first row:

$$
\gamma\varrho^2
\left|
\begin{array}{cccccc|ccccc}
(s+2\alpha) & 0 & \cdots & 0 & 0 & 0 & \varrho & 0 & \cdots & 0 & 0 \\
0 & (s+2\alpha) & \cdots & \vdots & \vdots & \vdots & -\varrho & \varrho & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & (s+2\alpha) & 0 & 0 & 0 & \cdots & -\varrho & \varrho \\
\gamma & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\gamma & \gamma & \cdots & \vdots & \vdots & \vdots & s & 0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & -\gamma & \gamma & 0 & \vdots & \vdots & \cdots & s & 0 \\
0 & 0 & \cdots & 0 & -\gamma & \gamma & 0 & 0 & \cdots & 0 & s
\end{array}
\right| = 0
$$

Again along column–$(N-1)$

$$
\gamma^2\varrho^2
\left|
\begin{array}{cccccc|ccccc}
(s+2\alpha) & 0 & \cdots & 0 & 0 & \varrho & 0 & \cdots & 0 & 0 \\
0 & (s+2\alpha) & \cdots & \vdots & \vdots & -\varrho & \varrho & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & (s+2\alpha) & 0 & 0 & \cdots & -\varrho & \varrho \\
\gamma & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\gamma & \gamma & \cdots & \vdots & \vdots & s & 0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\gamma & \gamma & 0 & 0 & \cdots & s & 0
\end{array}
\right| = 0
$$

Therefore, we find a pattern. To write the pattern, we define:

$$
\begin{aligned}
\tilde{I}_N &= \text{identity matrix whose first row and column–}N\text{ are all zeros} \\
\bar{A}_{12} &= \text{matrix } A_{12} \text{ without the last row} \\
\bar{A}_{12_N} &= \text{matrix of order } N \text{ and with the same pattern as } \bar{A}_{12} \\
\bar{A}_{21} &= \text{matrix } A_{21} \text{ without the first column} \\
\bar{A}_{21_N} &= \text{matrix of order } N \text{ and with the same pattern as } \bar{A}_{21}
\end{aligned}
$$

with this notation, we can write the determinant (88) as:

$$\text{with } i = 0, \text{it becomes} \quad \begin{vmatrix} s\tilde{I}_{N-i} & \bar{A}_{12_{N-i}} \\ \bar{A}_{21_{N-i}} & (s+2\alpha)I_{N-i} \end{vmatrix}$$

Define an iteration as:

1. Develop the determinant in cofactors along the first row

2. Develop the determinant in cofactors along column $(N-i)$

3. Switch the first row of blocks with the second one

4. Switch the first column of blocks with the second one

Then, we obtain:

$$\gamma^{i+1}\varrho^{i+1} \begin{vmatrix} (s+2\alpha)\tilde{I}_{N-(i+1)} & \bar{A}_{21_{N-(i+1)}} \\ \bar{A}_{12_{N-(i+1)}} & sI_{N-(i+1)} \end{vmatrix}$$

Iterate again and obtain:

$$\gamma^{i+2}\varrho^{i+2} \begin{vmatrix} s\tilde{I}_{N-(i+2)} & \bar{A}_{12_{N-(i+2)}} \\ \bar{A}_{21_{N-(i+2)}} & (s+2\alpha)I_{N-(i+2)} \end{vmatrix}$$

Also, considering $N = 1$ we write (88) as:

$$\begin{vmatrix} s & 0 & -\gamma & 0 \\ 0 & s & \gamma & 1 \\ \varrho & -\varrho & (s-\alpha) & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \tag{89}$$

Similarly to what we have done for the general case, we develop the determinant first along the last column

$$\begin{vmatrix} s & 0 & -\gamma \\ \varrho & \varrho & (s-\alpha) \\ 1 & 0 & 0 \end{vmatrix}$$

and next along the last row:

$$\begin{vmatrix} 0 & -\gamma \\ \varrho & (s-\alpha) \end{vmatrix} = -\varrho\gamma \neq 0,$$

and the proof that $G_{12}(s)$ has no zeros is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

$G_{12}(s)$ is a rational function whose poles are all the eigenvalues of A, since this transfer functions has no zeros. Therefore, we can write:

Corollary 7 $G_{12}(s)$ *is given by*

$$G_{12}(s) = \frac{K_G}{s\left(\dfrac{s^2}{\lambda_k \lambda_{-k}} + s\left(\dfrac{1}{\lambda_k} + \dfrac{1}{\lambda_{-k}}\right) + 1\right)} \tag{90}$$

where $\lambda_{\pm k}$ is given by (26).

5 Approximated Transfer functions

In this section we propose some approximations for the models of the transfer functions.

From Corollaries 1–2 and Theorem 3, $G_{ij}(s)$, $i,\ j = 1,\ 2$, are meromorphic functions given by

$$
\begin{aligned}
G_{11}(s) &= \frac{K_G}{s} \prod_{k=1}^{\infty} K_k \frac{(s+\alpha)^2 + (2k-1)^2\omega_0^2 - \alpha^2}{(s+\alpha)^2 + 4k^2\omega_0^2 - \alpha^2} \\
G_{21}(s) &= \frac{K_G}{s} \prod_{k=1}^{\infty} \frac{4k^2\omega_0^2}{(s+\alpha)^2 + 4k^2\omega_0^2 - \alpha^2} \\
G_{22}(s) &= -G_{11}(s) \\
G_{12}(s) &= -G_{21}(s)
\end{aligned}
\tag{91}
$$

where

$$\omega_0 = \frac{\pi}{2T_d} \tag{92}$$

and

$$K_k = \left(\frac{2k}{2k-1}\right)^2 \tag{93}$$

Natural gas is highly pressurized in transportation networks in order to expedite its flow. To ensure this, it must compressed periodically along the pipe. This is accomplished by compressor stations, which are usually placed at 60 Km to 250 Km intervals along the pipeline. As a result, the frequency ω_0 always remains much greater than α. Taking this into account as well as the requirement that its factors have a DC gain set to 1, we define $\widehat{K}_k$, and thence can approximate $G_{ij}(s)$, $i, j = 1, 2$ by

$$
\begin{aligned}
\widehat{G}_{11}(s) &= \frac{K_G}{s} \prod_{k=1}^{\infty} \widehat{K}_k \frac{(s+\alpha)^2 + (2k-1)^2\omega_0^2}{(s+\alpha)^2 + 4k^2\omega_0^2} \\
\widehat{G}_{21}(s) &= \frac{K_G}{s} \prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2\omega_0^2}{(s+\alpha)^2 + 4k^2\omega_0^2} \\
\widehat{G}_{12}(s) &= -\widehat{G}_{21}(s) \\
\widehat{G}_{22}(s) &= -\widehat{G}_{11}(s)
\end{aligned}
\tag{94}
$$

with

$$\widehat{K}_k = \frac{\alpha^2 + 4k^2\omega_0^2}{\alpha^2 + (2k-1)^2\omega_0^2}. \tag{95}$$

If we define $S = s + \alpha$ we can write

$$\widehat{G}_{11}(s) = \bar{G}_{11}(S) = \frac{K_G}{S - \alpha} \prod_{k=1}^{\infty} \widehat{K}_k \frac{S^2 + (2k-1)^2 \omega_0^2}{S^2 + 4k^2 \omega_0^2}$$

$$\widehat{G}_{21}(s) = \bar{G}_{21}(S) = \frac{K_G}{S - \alpha} \prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2 \omega_0^2}{S^2 + 4k^2 \omega_0^2}.$$

(96)

Theorem 10 considers auxiliary functions that lead to significant simplification in the representation of $\bar{G}_{11}(S)$ and $\bar{G}_{21}(S)$. However, before stating Theorem 10 we need to prove some intermediate results:

Theorem 6 *The function*

$$f(s) = \frac{e^{-sT_d}}{1 - e^{-2sTd}}$$

(97)

be expanded as

$$f(s) = \sum_{k=-\infty}^{\infty} \frac{a_k}{s - \lambda_k}$$

(98)

where a_k is the residual of $f(s)$ at $s = \lambda_k$, i. e.

$$a_k = \frac{(-1)^k}{2T_d}.$$

Since condition (129) holds (see appendix B) then the expansion (98) exits and the residuals a_k are given by
Proof:

$$\begin{aligned} a_k &= \frac{1}{2j\pi} \oint_{\mathcal{C}_k} f(s)ds = \lim_{s \to \lambda_k} (s - \lambda_k)f(s) = \lim_{s \to \lambda_k} \frac{(s - \lambda_k)e^{-Tds}}{1 - e^{-2T_ds}} \\ &= \lim_{s \to \lambda_k} \frac{e^{-sT_d} - (s - \lambda_k)T_d e^{-sT_d}}{2T_d e^{-2sT_d}} = \frac{e^{\lambda_k T_d}}{2T_d} = \frac{e^{\frac{kj\pi}{T_d}T_d}}{2Td} = \frac{e^{jk\pi}}{2T_d} = \frac{(-1)^k}{2T_d} \end{aligned}$$

$\square$

Theorem 7 *The function*

(99)

$$g(s) = \frac{\displaystyle\prod_{k=-\infty, k \neq 0}^{\infty} \left(-jk\frac{\pi}{T_d}\right)}{\displaystyle\prod_{k=-\infty}^{\infty} (s - jk\frac{\pi}{T_d})}.$$

can be written as:

$$2T_d \frac{e^{-T_d s}}{1 - e^{-2T_d s}} = 2T_d f(s).$$

Proof: Since $g(s)$ is proper it can be expanded as a Laurent series

$$g(s) = \sum_{k=-\infty}^{\infty} \frac{b_k}{s - \lambda_k}$$

where b_k is the residual of $g(s)$ at $s = \lambda_k = jk\frac{\pi}{T_d}$. In order to compute this residual we rewrite $g(s)$ as

$$g(s) = \lim_{M \to \infty} \frac{\prod\limits_{k=-M,k\neq 0}^{M} \left(-jk\frac{\pi}{T_d}\right)}{\prod\limits_{k=-M}^{M} \left(s - jk\frac{\pi}{T_d}\right)}.$$

The residual b_k is then given by

$$b_k = \frac{1}{2j\pi} \oint_{\mathcal{C}_k} g(s)ds = \lim_{s \to \lambda_k} (s - \lambda_k)g(s) = \lim_{M \to \infty} \frac{\prod\limits_{m=-M,m\neq 0}^{M} \left(jm\frac{\pi}{T_d}\right)}{\prod\limits_{m=-M}^{k-1} \left(j(k-m)\frac{\pi}{T_d}\right) \prod\limits_{m=k+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right)}$$

Given that

$$\left(-jm\frac{\pi}{T_d}\right)\left(jm\frac{\pi}{T_d}\right) = -j^2\left(\frac{m\pi}{T_d}\right)^2 = \left(\frac{m\pi}{T_d}\right)^2$$

we can rewrite b_k as

$$b_k = \lim_{M \to \infty} \frac{\prod\limits_{m=1}^{M} \left(m\frac{\pi}{T_d}\right)^2}{\prod\limits_{m=-M}^{k-1} \left(j(k-m)\frac{\pi}{T_d}\right) \prod\limits_{m=k+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right)}.$$

If now we replace define $\ell = k - m$ we get

$$
\begin{aligned}
b_k &= \lim_{M\to\infty} \frac{\displaystyle\prod_{m=1}^{M}\left(m\frac{\pi}{T_d}\right)^2}{\displaystyle\prod_{\ell=1}^{k+M}\left(j\ell\frac{\pi}{T_d}\right)\prod_{\ell=-(M-k)}^{-1}\left(j\ell\frac{\pi}{T_d}\right)} = \lim_{M\to\infty}\frac{\displaystyle\prod_{m=1}^{M}\left(m\frac{\pi}{T_d}\right)^2}{j^{2M}\displaystyle\prod_{\ell=1}^{k+M}\left(\ell\frac{\pi}{T_d}\right)\prod_{\ell=-(M-k)}^{-1}\left(\ell\frac{\pi}{T_d}\right)} \\[2ex]
&= \lim_{M\to\infty}\frac{\displaystyle\prod_{m=1}^{M}\left(m\frac{\pi}{T_d}\right)^2}{(-1)^M\displaystyle\prod_{\ell=1}^{k+M}\left(\ell\frac{\pi}{T_d}\right)\prod_{\ell=1}^{M-k}\left(-\ell\frac{\pi}{T_d}\right)} = \\[2ex]
&= \lim_{M\to\infty}\frac{\displaystyle\prod_{m=1}^{M}\left(m\frac{\pi}{T_d}\right)^2}{(-1)^M(-1)^{M-k}\displaystyle\prod_{\ell=1}^{k+M}\left(\ell\frac{\pi}{T_d}\right)\prod_{\ell=1}^{M-k}\left(\ell\frac{\pi}{T_d}\right)} \\[2ex]
&= \lim_{M\to\infty}\frac{\displaystyle\prod_{\ell=1}^{M}\left(\ell\frac{\pi}{T_d}\right)\prod_{\ell=1}^{M}\left(\ell\frac{\pi}{T_d}\right)}{(-1)^{2M-k}\displaystyle\prod_{\ell=1}^{k+M}\left(\ell\frac{\pi}{T_d}\right)\prod_{\ell=1}^{M-k}\left(\ell\frac{\pi}{T_d}\right)} = \lim_{M\to\infty}\frac{\displaystyle\prod_{\ell=M-k+1}^{M}\left(\ell\frac{\pi}{T_d}\right)}{(-1)^{-k}\displaystyle\prod_{\ell=M+1}^{M+k}\left(\ell\frac{\pi}{T_d}\right)} \\[2ex]
&= \lim_{M\to\infty}(-1)^k\frac{[M-(k-1)]\,[M-(k-2)]\ldots(M-1)M}{(M+1)(M+2)\ldots[M+(k-1)]\,(M+k)} \\[2ex]
&= \lim_{M\to\infty}(-1)^k\frac{M-(k-1)}{M+k}\frac{M-(k-2)}{M+(k-1)}\cdots\frac{M-1}{M+2}\frac{M}{M+1} \\[2ex]
&= (-1)^k\lim_{M\to\infty}\frac{M-(k-1)}{M+k}\lim_{M\to\infty}\frac{M-(k-2)}{M+(k-1)}\cdots\lim_{M\to\infty}\frac{M-1}{M+2}\lim_{M\to\infty}\frac{M}{M+1} = \\[2ex]
&= (-1)^k\prod_{\ell=1}^{k}\lim_{M\to\infty}\frac{M-(\ell-1)}{M+\ell}
\end{aligned}
$$

Since we can always make M infinitely greater than k then

$$
\lim_{M\to\infty}\frac{M-(\ell-1)}{M+\ell} = 1, \quad \forall\ell = 1,\ldots,k
$$

and, consequently,

$$
b_k = (-1)^k = 2T_d a_k
$$

and we conclude that

$$
g(s) = \frac{\displaystyle\prod_{k=-\infty,k\neq 0}^{\infty}\left(-jk\frac{\pi}{T_d}\right)}{\displaystyle\prod_{k=-\infty}^{\infty}\left(s - jk\frac{\pi}{T_d}\right)} = 2T_d\underbrace{\frac{e^{-T_d s}}{1 - e^{-2T_d s}}}_{:=f(s)}
$$

and the proof that the expansion of function $f(s)$ in Laurent series is possible is done in Appendix B.

□

Theorem 8 *The function*

$$v(s) = \frac{e^{-sT_d}}{1 + e^{-2sTd}} \qquad (100)$$

can expanded as

$$v(s) = \sum_{k=-\infty}^{\infty} \frac{c_k}{s - \lambda_k}$$

where c_k is the residual of $v(s)$ at $s = \lambda_k$:

$$c_k = \frac{j(-1)^{(k+1)}}{2T_d}$$

Proof: This function has poles at

$$\lambda_k = j(2k-1)\frac{\pi}{2T_d}, \quad k = -\infty, \ldots, -1, 0, 1, \infty$$

We can prove that condition (129) holds for $v(s)$ exactly the same way we did for $f(s)$. So, we can expand $v(s)$ as

$$v(s) = \sum_{k=-\infty}^{\infty} \frac{c_k}{s - \lambda_k}$$

where c_k is the residual of $v(s)$ at $s = \lambda_k$, i. e.

$$
\begin{aligned}
c_k &= \frac{1}{2j\pi} \oint_{\mathcal{C}_k} v(s)ds = \lim_{s \to \lambda_k}(s - \lambda_k)v(s) = \lim_{s \to \lambda_k} \frac{(s - \lambda_k)e^{-T_d s}}{1 + e^{-2T_d s}} \\
&= \lim_{s \to \lambda_k} \frac{e^{-sT_d} - (s - \lambda_k)T_d e^{-sT_d}}{-2T_d e^{-2sT_d}} = \frac{e^{\lambda_k T_d}}{2T_d} = \frac{e^{\frac{j(2k-1)\pi}{2T_d}T_d}}{2Td} = \frac{e^{j(2k+1)\frac{\pi}{2}}}{2T_d} = \frac{j(-1)^{(k+1)}}{2T_d}
\end{aligned}
$$

□

Theorem 9 *The function*

$$w(s) = \frac{\displaystyle\prod_{k=-\infty, k\neq0}^{\infty}\left(-j(2k-1)\frac{\pi}{2T_d}\right)}{\displaystyle\prod_{k=-\infty}^{\infty}\left(s - j(2k-1)\frac{\pi}{2T_d}\right)} \qquad (101)$$

can be written as:

$$w(s) = 2\frac{e^{-T_d s}}{1 + e^{-2T_d s}} = 2v(s).$$

Proof:

Since $w(s)$ is proper it can be expanded as

$$w(s) = \sum_{k=-\infty}^{\infty} \frac{d_k}{s - \lambda_k}$$

where d_k is the residual of $w(s)$ at $s = \lambda_k = j(2k-1)\frac{\pi}{2T_d}$. In order to compute this residual we rewrite $w(s)$ as

$$w(s) = \lim_{M\to\infty} \frac{\displaystyle\prod_{k=-M+1}^{M} \left(-j(2k-1)\frac{\pi}{2T_d}\right)}{\displaystyle\prod_{k=-M+1}^{M} \left(s - j(2k-1)\frac{\pi}{2T_d}\right)}.$$

The residual d_k is then given by

$$d_k = \frac{1}{2j\pi} \oint_{\mathcal{C}_k} w(s)ds = \lim_{s\to\lambda_k}(s-\lambda_k)w(s) = \lim_{M\to\infty} \frac{\displaystyle\prod_{m=-M+1}^{M} \left(j(2m-1)\frac{\pi}{2T_d}\right)}{\displaystyle\prod_{m=-M+1}^{k-1} \left(j(k-m)\frac{\pi}{T_d}\right) \prod_{m=k+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right)}$$

Given that the numerator can be expressed as the product of two complex factors we can write d_k as

$$d_k = \lim_{M\to\infty} \frac{\displaystyle\prod_{m=1}^{M} \left((2m-1)\frac{\pi}{2T_d}\right)^2}{\displaystyle\prod_{m=-M+1}^{k-1} \left(j(k-m)\frac{\pi}{T_d}\right) \prod_{m=k+1}^{M} \left(j(k-m)\frac{\pi}{T_d}\right)}.$$

If now we replace define $\ell = k - m$ we get

$$d_k = \lim_{M\to\infty} \frac{\displaystyle\prod_{m=1}^{M} \left((2m-1)\frac{\pi}{2T_d}\right)^2}{\displaystyle\prod_{\ell=1}^{k+M-1} \left(j\ell\frac{\pi}{T_d}\right) \prod_{\ell=-(M-k)}^{-1} \left(j\ell\frac{\pi}{T_d}\right)} = \lim_{M\to\infty} \frac{\displaystyle\prod_{m=1}^{M} \left((2m-1)\frac{\pi}{2T_d}\right)^2}{j^{2M-1}\displaystyle\prod_{\ell=1}^{k+M-1} \left(\ell\frac{\pi}{T_d}\right) \prod_{\ell=-(M-k)}^{-1} \left(\ell\frac{\pi}{T_d}\right)} =$$

$$= \lim_{M\to\infty} \frac{\displaystyle\prod_{m=1}^{M} \left((2m-1)\frac{\pi}{2T_d}\right)^2}{(-j)^{2M}j^{-1}\displaystyle\prod_{\ell=1}^{k+M-1} \left(\ell\frac{\pi}{T_d}\right) \prod_{\ell=1}^{M-k} \left(-\ell\frac{\pi}{T_d}\right)} =$$

$$= \lim_{M\to\infty} \frac{j\displaystyle\prod_{m=1}^{M} \left((2m-1)\frac{\pi}{2T_d}\right)^2}{(-1)^M(-1)^{M-k}\displaystyle\prod_{\ell=1}^{k+M-1} \left(\ell\frac{\pi}{T_d}\right) \prod_{\ell=1}^{M-k} \left(\ell\frac{\pi}{T_d}\right)} = \frac{-j}{(-1)^{2M-k}T_d} = \frac{j(-1)^{k+1}}{T_d}$$

and we conclude that

$$w(s) = \frac{\displaystyle\prod_{k=-\infty,k\neq0}^{\infty}\left(-j(2k+1)\frac{\pi}{2T_d}\right)}{\displaystyle\prod_{k=-\infty}^{\infty}(s-j(2k+1)\frac{\pi}{T_d})} = 2\frac{e^{-T_d s}}{1+e^{-2T_d s}} = 2v(s)$$

$\square$

Theorem 10 *Consider the following functions*

$$G_e(S) = \frac{\displaystyle\prod_{k=-\infty,k\neq0}^{\infty}\left(-jk\frac{\pi}{T_d}\right)}{\displaystyle\prod_{k=-\infty}^{\infty}(S-jk\frac{\pi}{T_d})} \qquad (102)$$

$$G_o(S) = \frac{\displaystyle\prod_{k=-\infty}^{\infty}\left(-j(2k-1)\frac{\pi}{2T_d}\right)}{\displaystyle\prod_{k=-\infty}^{\infty}\left(S-j(2k-1)\frac{\pi}{2T_d}\right)} \qquad (103)$$

$$F_e(s) = \frac{e^{-T_d S}}{1-e^{-2T_d S}} \qquad (104)$$

$$F_o(s) = \frac{e^{-ST_d}}{1+e^{-2ST_d}} \qquad (105)$$

then

$$G_e(S) = 2T_d F_e(S) \qquad (106)$$
$$G_o(S) = 2F_o(S) \qquad (107)$$

Proof: To prove the results in Theorem 7 we consider $g(s) = G_e(s)$ and $f(s) = F_e(S)$ and also in Theorem 9 we consider $w(s) = G_o(s)$ and $v(s) = F_o(S)$.

$\square$

$G_e(S)$ and $G_o(S)$ may also be written as

$$G_e(S) = \frac{\displaystyle\prod_{k=1}^{\infty}4k^2\omega_0^2}{\displaystyle\prod_{k=0}^{\infty}(S^2+4k^2\omega_0^2)} \qquad (108)$$

$$G_o(S) = \frac{\displaystyle\prod_{k=1}^{\infty}(2k-1)^2\omega_0^2}{\displaystyle\prod_{k=1}^{\infty}(S^2+(2k-1)^2\omega_0^2)} \qquad (109)$$

where ω_o is defined in equation (92). As a result,

$$\prod_{k=1}^{\infty} \frac{S^2 + (2k-1)^2\omega_0^2}{S^2 + 4k^2\omega_0^2} = \frac{SG_e(S)}{G_o(S)} \prod_{k=1}^{\infty} \frac{1}{K_k}$$

$$\prod_{k=1}^{\infty} \frac{1}{S^2 + 4k^2\omega_0^2} = SG_e(S) \prod_{K=1}^{\infty} \frac{1}{4k^2\omega_0^2}$$

with K_k being defined in equation (93). Now, using these equations and theorem 10 we can write

$$\prod_{k=1}^{\infty} \widehat{K}_k \frac{S^2 + (2k-1)^2\omega_0^2}{S^2 + 4k^2\omega_0^2} = \bar{K}_{11} \frac{SG_e(S)}{G_o(S)} =$$
$$= \bar{K}_{11}T_d \frac{SF_e(S)}{F_o(S)} \tag{110}$$
$$\prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2\omega_0^2}{S^2 + 4k^2\omega_0^2} = \bar{K}_{12}F_e(S) = 2\bar{K}_{12}T_dG_e(S)$$

with

$$\bar{K}_{11} = \prod_{k=1}^{\infty} \frac{\widehat{K}_k}{K_k}$$
$$\bar{K}_{12} = \prod_{k=1}^{\infty} \frac{\alpha^2 + 4k^2\omega_0^2}{4k^2\omega_0^2} \tag{111}$$

we thus can rewrite $\bar{G}_{11}(S)$ and $\bar{G}_{21}(S)$ as

$$\bar{G}_{11}(S) = \frac{K_GT_d\bar{K}_{11}S\left(1 + e^{-2ST_d}\right)}{(S-\alpha)\left(1 - e^{-2ST_d}\right)}$$
$$\bar{G}_{21}(S) = \frac{2K_GT_d\bar{K}_{12}Se^{-ST_d}}{(S-\alpha)\left(1 - e^{-2ST_d}\right)}. \tag{112}$$

We can compute $\widehat{G}_{11}(s)$ and $\widehat{G}_{21}(s)$ from these equations by replacing S with $s + \alpha$:

$$\widehat{G}_{11}(s) = \frac{K_GT_d\bar{K}_{11}(s+\alpha)\left(1 + e^{-2\alpha T_d}e^{-2sT_d}\right)}{s\left(1 - e^{-2\alpha T_d}e^{-2sT_d}\right)}$$
$$\widehat{G}_{21}(s) = \frac{2K_GT_d\bar{K}_{12}(s+\alpha)e^{-\alpha T_d}e^{-sT_d}}{s\left(1 - e^{-2\alpha T_d}e^{-2sT_d}\right)}. \tag{113}$$

Given that $\widehat{G}_{11}(s)$ and $\widehat{G}_{21}(s)$ were defined in such a way that

$$\lim_{s\to 0} s\widehat{G}_{11}(s) = \lim_{s\to 0} s\widehat{G}_{21}(s) = \lim_{s\to 0} sG_{11}(s)$$
$$= \lim_{s\to 0} sG_{21}(s) = K_G, \tag{114}$$

we can rewrite (113) as

$$\widehat{G}_{11}(s) = K_{11} \frac{(s+\alpha)\left(1 + e^{-2\alpha T_d}e^{-2sT_d}\right)}{s\left(1 - e^{-2\alpha T_d}e^{-2sT_d}\right)}$$
$$\widehat{G}_{21}(s) = K_{21} \frac{(s+\alpha)e^{-sT_d}}{s\left(1 - e^{-2\alpha T_d}e^{-2sT_d}\right)}. \tag{115}$$

with

$$K_{11} = \frac{K_G \left(1 - e^{-2\alpha T_d}\right)}{\alpha \left(1 + e^{-2\alpha T_d}\right)}$$

$$K_{21} = \frac{K_G \left(1 - e^{-2\alpha T_d}\right)}{\alpha}.$$

(116)

6 Case study

In this section we study a small pipeline using the lumped linear model derived above. The pipe has a length $L = 35$Km and a diameter $\mathcal{D} = 793$mm. The friction factor is $f_c = 0.0079$ and the isothermal speed of sound is $c = 300$m/sec. We considered $q_m = 90Kg/sec$ and $p_m = 80$ bar $= 8 \times 10^6$ Pascal as the mass-flow and pressure nominal values. The α and K_G parameters were calculated from equations (3) and Corollary 5, respectively, and the values of 0.0051 and 5.064 were obtained. We computed the frequency responses (FR) of $G_{11}(s)$ and $G_{21}(s)$ from equations (91) using truncated approximations of order n, $G_{11}^{(n)}(s)$ and $G_{11}^{(n)}(s)$, respectively, in a frequency bandwidth, $BW = \left[10^{-4}\text{rad/sec}, \ 1 \ \text{rad/sec}\right]$.
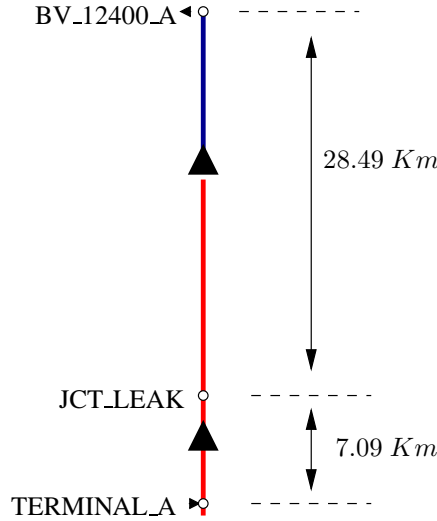
Figure 1: Gas pipeline topology.

Figures 2 and 3 display the respective Bode diagrams and compare them with their approximations $\widehat{G}_{11}(s)$ and $\widehat{G}_{21}(s)$. $G_{11}(s)$ converges very fast. The FR of a truncated approximation with two hundred poles and zeros had already converged to its limit in the whole frequency interval BW. Figure 2 shows that there are no significant differences between $G_{11}(s)$ and $\widehat{G}_{11}(s)$. Consequently, $G_{11}(s)$ can be substituted by its approximation without loss of accuracy and with a significant reduction of the computational costs.

The convergence of $G_{21}(s)$ is much slower. A two thousand order approximation didn't converge in the whole bandwidth BW. We can subsequently conclude that a truncated approximation of equation (91) needs too many factors leading, therefore, to high order transfer functions with high computational costs.

However, from Figures 3 and 4 one can conclude that $G_{21}(s)$ can be also substituted by its approximation. In Figure 4 the Bode diagrams of $\widehat{G}_{21}(s)$ and $G_{21}^{(2000)}(s)$ are compared with a better resolution, i.e. the phase is restricted to the range $(-180$ degrees, 180 degrees$)$. The phase discontinuities are due to the phase-crossing
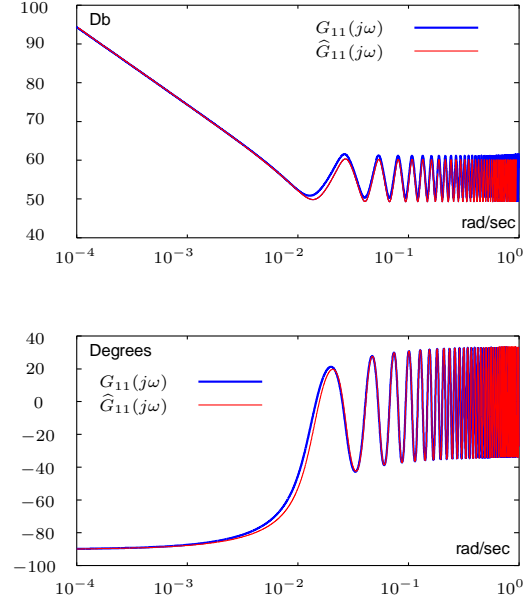
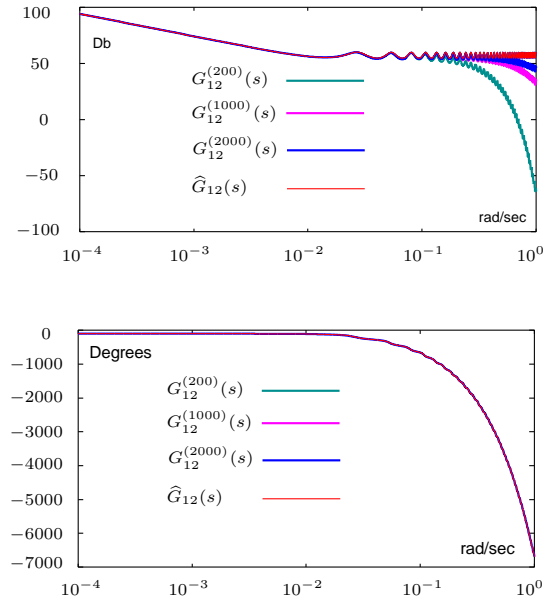Figure 2: Bode plots of $G_{11}(s)$ and the approximation $\widehat{G}_{11}(s)$.

Figure 3: Bode plots of both $G_{12}^{(n)}(s)$, $n = 200,\ 100,\ 2000$ and the approximation $\widehat{G}_{12}(s)$.
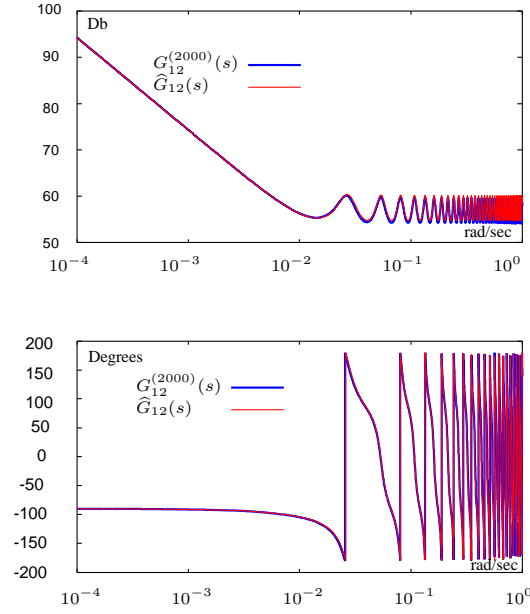
Figure 4: Higher resolution Bode plots of both $G_{12}^{(2000)}(s)$ and the approximation $\widehat{G}_{12}(s)$.
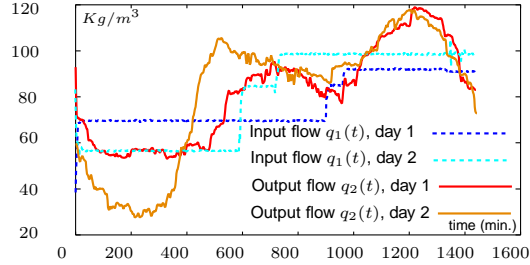

Figure 5: Input and output mass-flows on the both days.

of the odd multiples of 180 degrees which are converted from -180 degrees to 180 degrees. Also notice that the Bode diagram of the finite approximation $G_{12}^{(2000)}(s)$ converges to $G_{12}(s)$ almost in the whole frequency BW.

This pipeline was simulated taking two normal days operation data as the input and output mass-flows. The simulation was performed with the previous referred SIMONE® simulator. Figure 5 shows the input and output mass-flows on both days

The intake and offtake massflows are denoted by $q_1(t)$ and $q_2(t)$, respectively.

Figures 6 and 7 compare the input and output pressures simulated with the lumped transfer function model

$$
\begin{aligned}
P_1(s) &= \widehat{G}_{11}(s)Q_1(s) - \widehat{G}_{21}(s)Q_2(s) \\
P_2(s) &= \widehat{G}_{21}(s)Q_1(s) - \widehat{G}_{22}(s)Q_2(s)
\end{aligned}
\tag{117}
$$

with the ones simulated with SIMONE® for the two days. The intake and offtake pressures are denoted by $p_1(t)$ and $p_2(t)$, respectively. We can see that the results are better for the first day. This is expectable, since the mass-flows and pressures of the second day data present stronger deviations from the nominal values used for the calculation of the α parameter. As a matter of fact, on the second day the quotient $(q_1(t) + q_2(t)) / (p_1(t) + p_2(t))$ has a root mean square deviation from the nominal value of about 50%
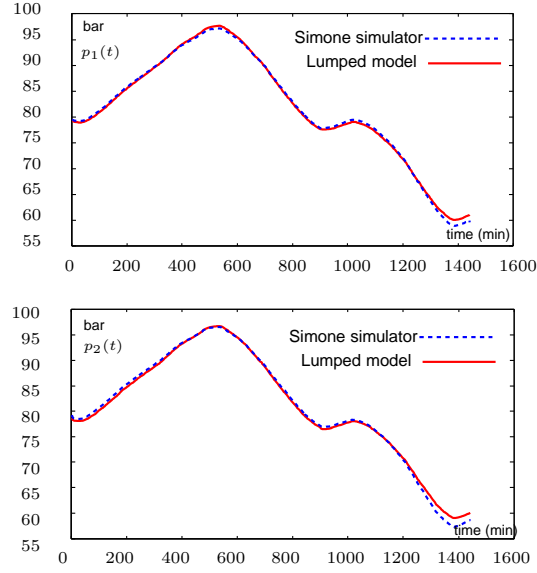
Figure 6: Input and Output pressures, P_1, and P_2, respectively, simulated by the lumped model (117) (solide red line) and by SIMONE® using the first day data.
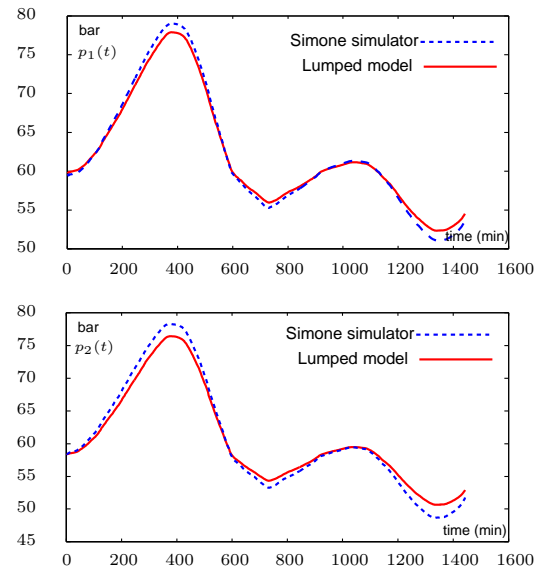

Figure 7: Input and Output pressures, P_1, and P_2, respectively, simulated by the lumped model (117) (solide red line) and by SIMONE® using the second day data.

(this deviation is of 30% in the first day). Yet, in both cases, the model has well captured the dynamics of the system and, therefore, it seems to be a valuable tool for gas leakage detection and gas networks controller design.

A   Change of variables in an integral model for a short gas pipeline

In this section, a change in the state-space variables of the integral model is performed. The purpose is to obtain a simpler system matrix in order to simplify the determination of the eigenvalues of the system.

Consider model (10) with the respective matrices defined according to (12)–(17).

Consider for this system the following change of variables:

$$
\begin{aligned}
z_2 &= x_2 + x_2 + \cdots + x_N + x_{N+1} \\
z_2 &= x_2 - x_2 \\
z_3 &= x_2 - x_3 \\
&\vdots \\
z_{N+1} &= x_N - x_{N+1} \\
z_{N+2} &= x_{N+2} \\
z_{N+3} &= x_{N+3} \\
&\vdots \\
z_{2N+1} &= x_{2N+1}
\end{aligned}
\tag{118}
$$

and we obtain the following realisation

$$
\begin{aligned}
\dot{z}_2(t) &= \frac{c^2}{\mathcal{A}\Delta\ell} u_2(t) - \frac{c^2}{\mathcal{A}\Delta\ell} u_2(t) \\[2mm]
\dot{z}_2(t) &= -2\frac{c^2}{\mathcal{A}\Delta\ell} z_{N+2}(t) + \frac{c^2}{\mathcal{A}\Delta\ell} z_{N+3}(t) \\[2mm]
\dot{z}_3(t) &= \frac{c^2}{\mathcal{A}\Delta\ell} z_{N+2}(t) - 2\frac{c^2}{\mathcal{A}\Delta\ell} z_{N+3}(t) + \frac{c^2}{\mathcal{A}\Delta\ell} z_{N+4}(t) \\[2mm]
&\vdots
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
\dot{z}_i(t) &= \frac{c^2}{\mathcal{A}\Delta\ell} z_{N+i-1}(t) - 2\frac{c^2}{\mathcal{A}\Delta\ell} z_{N+i}(t) + \frac{c^2}{\mathcal{A}\Delta\ell} z_{N+i+1}(t) \\[3mm]
\dot{z}_{N+1}(t) &= \frac{c^2}{\mathcal{A}\Delta\ell} z_{2N}(t) - 2\frac{c^2}{\mathcal{A}\Delta\ell} z_{2N+1}(t) + \frac{c^2}{\mathcal{A}\Delta\ell} u_2(t) \\[2mm]
\dot{z}_{N+2}(t) &= \frac{\mathcal{A}}{\Delta\ell} z_2(t) - \frac{f_c c^2 Q_{m1}}{2\mathcal{D}\mathcal{A}P_{m1}} z_{N+2}(t) \\[2mm]
\dot{z}_{N+3}(t) &= \frac{\mathcal{A}}{\Delta\ell} z_3(t) - \frac{f_c c^2 Q_{m1}}{2\mathcal{D}\mathcal{A}P_{m1}} z_{N+3}(t) \\[2mm]
&\vdots
\end{aligned}
\tag{120}
$$

$$\dot{z}_{N+i}(t) = \frac{\mathcal{A}}{\Delta\ell}z_i(t) - \frac{f_c c^2 Q_{m1}}{2\mathcal{D}\mathcal{A}P_{m1}}z_{N+i}(t)$$

$$\vdots \tag{121}$$

$$\dot{z}_{2N+1}(t) = \frac{\mathcal{A}}{\Delta\ell}z_{N+1}(t) - \frac{f_c c^2 Q_{m1}}{2\mathcal{D}\mathcal{A}P_{m1}}z_{2N+1}(t)$$

$$
\begin{aligned}
y_2(t) &= \frac{1}{N+1}z_2(t) + \frac{N}{N+1}z_2(t) + \frac{N-1}{N+1}z_3(t) + \cdots + \frac{1}{N+1}z_{N+1}(t) \\
y_2(t) &= \frac{1}{N+1}z_2(t) - \frac{1}{N+1}z_2(t) - \frac{2}{N+1}z_3(t) - \cdots - \frac{N}{N+1}z_{N+1}(t)
\end{aligned}
\tag{122}
$$

and in matricial form, we have:

$$
\begin{aligned}
\dot{z}(t) &= \bar{A}z(t) + \bar{B}u(t) \\
y(t) &= \bar{C}z(t)
\end{aligned}
\tag{123}
$$

with

$$
\bar{A} = \left[
\begin{array}{c|c|c}
0 & 0_{1\times N} & 0_{1\times N} \\
\hline
0_{N\times 1} & \bar{A}_{11} & \bar{A}_{12} \\
\hline
0_{N\times 1} & \bar{A}_{21} & \bar{A}_{22}
\end{array}
\right]
\tag{124}
$$

$$\bar{A}_{11} = 0_{N\times N} \tag{125}$$

$$
\bar{A}_{12} = \left[
\begin{array}{cccccc}
-2\dfrac{c^2}{\mathcal{A}\Delta\ell} & \dfrac{c^2}{\mathcal{A}\Delta\ell} & 0 & \cdots & 0 & 0 & 0 \\[2ex]
\dfrac{c^2}{\mathcal{A}\Delta\ell} & -2\dfrac{c^2}{\mathcal{A}\Delta\ell} & \dfrac{c^2}{\mathcal{A}\Delta\ell} & \cdots & \vdots & \vdots & \vdots \\[2ex]
0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -2\dfrac{c^2}{\mathcal{A}\Delta\ell} & \ddots & \vdots & \vdots & \vdots \\[2ex]
\vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots \\[2ex]
\vdots & \vdots & \vdots & \cdots & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -2\dfrac{c^2}{\mathcal{A}\Delta\ell} & \dfrac{c^2}{\mathcal{A}\Delta\ell} \\[2ex]
0 & 0 & 0 & \cdots & 0 & \dfrac{c^2}{\mathcal{A}\Delta\ell} & -2\dfrac{c^2}{\mathcal{A}\Delta\ell}
\end{array}
\right] \in \mathbb{R}^{N\times N} \tag{126}
$$

$$\bar{A}_{21} = \frac{\mathcal{A}}{\Delta\ell}I_N, \quad I_N - \text{identity matrix } N \times N$$

$$\bar{A}_{22} = -\frac{f_c c^2 Q_m}{2\mathcal{D}\mathcal{A}P_m}I_N$$

$$\bar{B} = \frac{c^2}{\mathcal{A}\Delta\ell}\left[\; e_2 \;\middle|\; -e_2 + e_{N+1} \;\right] \in \mathbb{R}^{(2N+1)\times 2} \tag{127}$$

$$\bar{C} = \begin{bmatrix} \dfrac{1}{N+1} & \dfrac{N}{N+1} & \dfrac{N-1}{N+1} & \cdots & \dfrac{1}{N+1} \\[2mm] \dfrac{1}{N+1} & -\dfrac{1}{N+1} & -\dfrac{2}{N+1} & \cdots & -\dfrac{N}{N+1} \end{bmatrix} \in \mathbb{R}^{2\times(2N+1)}$$

Matrices A and $\bar{A}$ have the same spectrum, however its calculation seems to be easier if we use matrix $\bar{A}$.

B Rational expansion of meromorphic functions

Let $f(s)$ be a function meromorphic in the finite complex plane with poles at $\lambda_1,\ \lambda_2,\ \ldots$, and let $(\Gamma_1,\ \Gamma_2,\ldots)$ be a sequence of simple closed curves such that:

- The origin lies inside each curve Γ_k.

- No curve passes through a pole of f.

- Γ_k lies inside Γ_{k+1} for all k

- $\lim\limits_{k\to\infty} d(\Gamma_k) = \infty$, where $d(\Gamma_k)$ gives the distance from the curve to the origin

Suppose also that there exists an integer p such that

$$\lim_{k\to\infty} \oint_{\Gamma_k} \left|\frac{f(s)}{s^p}\right| |ds| < \infty.$$

Denoting the principal part of the Laurent series of f about the point λ_k as $PP\left[f(s); s = \lambda_k\right]$, we have, if $p < 0$.

$$f(z) = \sum_{k=0}^{\infty} \mathrm{PP}(f(z); z = \lambda_k).$$

Theorem 11 *Consider function* $f(s) = \dfrac{e^{-T_d s}}{1 - e^{-2T_d s}}$. *There exists an integer p such that*

$$\lim_{k\to\infty} \oint_{\Gamma_k} \left|\frac{f(s)}{s^p}\right| |ds| < \infty. \tag{128}$$

Proof: This function has poles at

$$\lambda_k = j2k\frac{\pi}{2T_d} = jk\frac{\pi}{T_d}, \quad k = -\infty, \ldots, -1, 0, 1, \infty$$

The contours Γ_k will be squares vertices at $\pm(2k-1)\dfrac{\pi}{2T_d} \pm j(2k-1)\dfrac{\pi}{2T_d}$, $k > 1$, traversed counterclockwise, which are easily seen to satisfy the necessary conditions. To see what are the terms of the Laurent
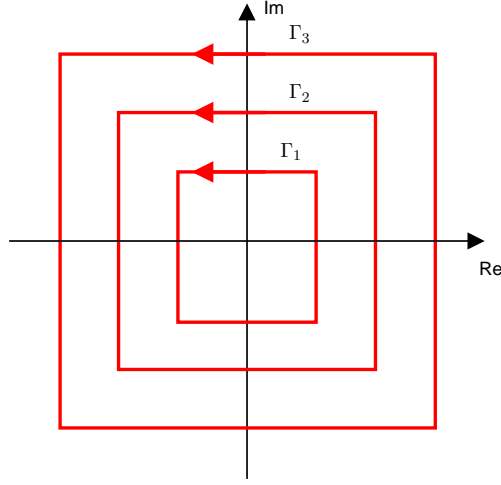
Figure 8: Contours

series expansion of $f(s)$ we need do see for which p the condition

$$\lim_{k \to \infty} \oint_{\Gamma_k} \left| \frac{f(s)}{s^p} \right| |ds| < \infty \tag{129}$$

holds. We can partition this integral as

$$\oint_{\Gamma_k} \left| \frac{f(s)}{s^p} \right| |ds| = \oint_{\Gamma_{k1}} \left| \frac{f(s)}{s^p} \right| |ds| + \oint_{\Gamma_{k2}} \left| \frac{f(s)}{s^p} \right| |ds| + \oint_{\Gamma_{k3}} \left| \frac{f(s)}{s^p} \right| |ds| + \oint_{\Gamma_{k4}} \left| \frac{f(s)}{s^p} \right| |ds|$$

where

$$\Gamma_{k1} = \left\{ s : s = (2k-1)\frac{\pi}{2T_d} + j\omega, \ \omega \uparrow, \ \omega \in [-(2k-1)\frac{\pi}{2T_d}, (2k-1)]\frac{\pi}{2T_d} \right\}$$

$$\Gamma_{k2} = \left\{ s : s = \sigma + j2(k-1)\frac{\pi}{2T_d}, \ \sigma \downarrow, \ \sigma \in [-(2k-1)\frac{\pi}{2T_d}, (2k-1)]\frac{\pi}{2T_d} \right\}$$

$$\Gamma_{k3} = \left\{ s : s = -(2k-1)\frac{\pi}{2T_d} + j\omega, \ \omega \downarrow, \ \omega \in [-(2k-1)\frac{\pi}{2T_d}, (2k-1)]\frac{\pi}{2T_d} \right\}$$

$$\Gamma_{k4} = \left\{ s : s = \sigma - j(2k-1)\frac{\pi}{2T_d}, \ \sigma \uparrow, \ \omega \in [-(2k-1)\frac{\pi}{2T_d}, (2k-1)]\frac{\pi}{2T_d} \right\}.$$

Notice that

- For $s \in \Gamma_{k1}$, $|ds| = d\omega$.

- For $s \in \Gamma_{k2}$, $|ds| = -d\sigma$.

- For $s \in \Gamma_{k3}$, $|ds| = -d\omega$.

- For $s \in \Gamma_{k4}$, $|ds| = d\sigma$.

Now we can write

$$\lim_{k\to\infty} \oint_{\Gamma_k} \left|\frac{f(s)}{s^p}\right| |ds| = \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)\right| d\omega - \int_{(2k-1)\frac{\pi}{2T_d}}^{-(2k-1)\frac{\pi}{2T_d}} \left|f\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)\right| d\sigma$$

$$-\int_{(2k-1)\frac{\pi}{2T_d}}^{-(2k-1)\frac{\pi}{2T_d}} \left|f\left(-(2k-1)\frac{\pi}{2T_d}+j\omega\right)\right| d\omega + \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left(\sigma-j(2k-1)\frac{\pi}{2T_d}\right)\right| d\sigma$$

$$= \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)\right| d\omega + \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)\right| d\sigma$$

$$+ \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left(-(2k-1)\frac{\pi}{2T_d}+j\omega\right)\right| d\omega + \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left(\sigma-j(2k-1)\frac{\pi}{2T_d}\right)\right| d\sigma$$

Next we analyze the four terms of this integral

First term

$$\int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)\right| d\omega = \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|\frac{e^{-(2k-1)\frac{\pi}{2}}e^{-j\omega T_d}}{\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)^p\left(1-e^{-(2k-1)\pi}e^{-j\omega T_d}\right)}\right| d\omega =$$

$$\int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|\frac{e^{-(2k-1)\frac{\pi}{2}}}{\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)^p\left(1-e^{-(2k-1)\pi}e^{-j\omega T_d}\right)}\right| d\omega.$$

Given that

$$\lim_{k\to\infty} e^{-(2k-1)\frac{\pi}{2}} = 0$$

then

$$\lim_{k\to\infty} \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)\right| d\omega = \lim_{k\to\infty} \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|\frac{e^{-(2k-1)\frac{\pi}{2}}}{\left((2k-1)\frac{\pi}{2T_d}+j\omega\right)^p\left(1-e^{-(2k-1)\pi}e^{-j\omega T_d}\right)}\right| d\omega = 0.$$

Second term

$$\int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|f\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)\right| d\sigma = \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left|\frac{e^{-\sigma T_d}e^{-j(2k-1)\frac{\pi}{2}}}{\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)^p\left(1-e^{-2\sigma T_d}e^{-j(2k-1)\pi}\right)}\right| d\sigma =$$

$$= \int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left|\frac{e^{-\sigma T_d}}{\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)^p\left(1-e^{-2\sigma T_d}(-1)^k\right)}\right| d\sigma + \int_{0}^{(2k-1)\frac{\pi}{2T_d}} \left|\frac{e^{-\sigma T_d}}{\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)^p\left(1-e^{-2\sigma T_d}(-1)^k\right)}\right| d\sigma =$$

Notice that

$$\int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left|\frac{e^{-\sigma T_d}}{\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)^p\left(1-e^{-2\sigma T_d}(-1)^k\right)}\right| d\sigma \leq \int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left|\frac{e^{-\sigma T_d}}{\left(\sigma+j(2k-1)\frac{\pi}{2T_d}\right)^p\left(1-e^{-2\sigma T_d}\right)}\right| d\sigma$$

On the other hand, for $\sigma < 0$

$$\left|1-e^{-\sigma T_d}\right| = e^{|\sigma|T_d} - 1 < e^{|\sigma T_d|} = e^{-\sigma T_d}.$$

As a result

$$\int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma < \int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p e^{-2\sigma T_d}} \right| d\sigma =$$

$$= \int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left| \frac{e^{\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p} \right| d\sigma$$

Making $k \to \infty$,

$$\lim_{k\to\infty} \int_{-(2k-1)\frac{\pi}{2T_d}}^{0} \left| \frac{e^{\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p} \right| d\sigma = M_1.^1$$

Notice also that

$$\int_{0}^{(2k-1)\frac{\pi}{2T_d}} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d}(-1)^k)} \right| d\sigma \leq \int_{0}^{(2k-1)\frac{\pi}{2T_d}} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma$$

For $\sigma > 0$

$$1 - e^{-\sigma T d} < 1.$$

Consequently

$$\int_{0}^{(2k-1)\frac{\pi}{2T_d}} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p (1 - e^{-2\sigma T_d})} \right| d\sigma < \int_{0}^{(2k-1)\frac{\pi}{2T_d}} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p} \right| d\sigma$$

Making $k \to \infty$ again

$$\lim_{k\to\infty} \int_{0}^{(2k-1)\frac{\pi}{2T_d}} \left| \frac{e^{-\sigma T_d}}{\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right)^p} \right| d\sigma = M_1$$

and we conclude that

$$\lim_{k\to\infty} \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left| f\left(\sigma + j(2k-1)\frac{\pi}{2T_d}\right) \right| d\sigma < 2M_1 < \infty.$$

Third term This term is similar to the first one and using the same arguments we can prove that

$$\lim_{k\to\infty} \int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left| f\left(-(2k-1)\frac{\pi}{2T_d} + j\omega\right) \right| d\omega = 0.$$

In similar way that we did for the second term we can prove that

$$\int_{-(2k-1)\frac{\pi}{2T_d}}^{(2k-1)\frac{\pi}{2T_d}} \left| f\left(\sigma - j(2k-1)\frac{\pi}{2T_d}\right) \right| d\sigma < 2M_1$$

We can now conclude that condition (129) holds for any $p < 0$.

$\square$

Since there exists an integer p such that

$$\lim_{k \to \infty} \oint_{\Gamma_k} \left| \frac{f(s)}{s^p} \right| |ds| < \infty.$$

Then, denoting the principal part of the Laurent series of f about the point λ_k as $PP\left[f(s); s = \lambda_k\right]$, we have, if $p < 0$.

$$f(z) = \sum_{k=0}^{\infty} \mathrm{PP}(f(z); z = \lambda_k).$$

References

[1] D. S. Bernstein, *Matrix Mathematics - Theory, Facts, and Formulas with Application to Linear System Theory*, Princeton University Press, Princeton NJ; 2005.

[2] J. L. Martins de Carvalho, *Dynamical Systems and Automatic Control*, Prentice Hall, London; 1999.

[3] Simone Research Group and Liwacom, *Simone software: equations and methods*, Simone Research Group and LIWACOM, Germany, 2004.

[4] W-C Yueh, "Eigenvalues of Several Tridiagonal Matrices", in *Applied Mathematics E-Notes*, Vol 5, pp. 72, 2005.